# General Information

- Textbook: *Introduction to Real Analysis* by Robert G. Bartle, Donald R. Sherbert. (Try to google the title of the textbook for MORE information!)
- This course is a continuation of the course MATH2050 Mathematical Analysis I, the materials depend heavily on the previous course, please make sure that you are familiar with them.
- I am the tutor of the tutorial sessions. You may call me **Ernest**. My office is located at **LSB G06** and my office hour for this course is **Wednesday 12:30-2:30**. You may come to me during this session if you need any help. My email address is **ylfan@math.cuhk.edu.hk**. You are welcomed to send me an email if you need help.
- Please visit the course web-page at `https://www.math.cuhk.edu.hk/course/1920/math2060b` frequently to get the most updated information. It shall contain the information for the Homework and Quizzes, as well as lecture notes and tutorial notes.
- There are two tutorial sessions per week. You may attend **EITHER ONE** session. The materials covered in each session will be the same.

# Uniform Continuity

First of all, let's recall the definitions of different types of continuity and compare them.

**Definition** (c.f. Definition 5.4.1)**.** Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function.

(a) $f$ is said to be **continuous at** $x \in A$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that
$$|f(x) - f(u)| < \varepsilon \quad \text{whenever } u \in A \text{ and } |x - u| < \delta.$$

(b) $f$ is said to be **continuous on** $A$ if $f$ is continuous at every $x \in A$. i.e., for any $x \in A$ and $\varepsilon > 0$, there exists $\delta > 0$ such that
$$|f(x) - f(u)| < \varepsilon \quad \text{whenever } u \in A \text{ and } |x - u| < \delta.$$

(c) $f$ is said to be **uniformly continuous on** $A$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that
$$|f(x) - f(u)| < \varepsilon \qquad \text{whenever } x, u \in A \text{ and } |x - u| < \delta.$$

**Remark.** Notice that the uniform continuity on a set implies the continuity on the same set. i.e., uniform continuity is a stronger continuity. Try to prove it!

**Exercise 1.** Determine whether the following functions are uniformly continuous on its domain.

(a) $f(x) = x$ on $\mathbb{R}$.

(b) $f(x) = x^2$ on $\mathbb{R}$.

(c) $f(x) = 1/x$ on $(0, \infty)$.

(d) $f(x) = 1/(1 + x^2)$ on $\mathbb{R}$.

**Solution.** (a) It is uniformly continuous on $\mathbb{R}$. Just take $\varepsilon = \delta$.

(b) It is not uniformly continuous on $\mathbb{R}$. To see this, we need to choose an $\varepsilon > 0$ and show that for every $\delta > 0$, there are two real numbers $x$ and $u$ such that

$$|x - u| < \delta \quad \text{and} \quad |f(x) - f(u)| \geq \varepsilon.$$

Take $\varepsilon = 1$. For every $\delta > 0$, take $x = 1/\delta + \delta/2$ and $u = 1/\delta$. Then $|x - u| < \delta$ and

$$|f(x) - f(u)| = \left| \left( \frac{1}{\delta} + \frac{\delta}{2} \right) - \left( \frac{1}{\delta} \right)^2 \right| = 1 + \frac{\delta^2}{4} \geq 1.$$

(c) It is not uniformly continuous on $(0, \infty)$. Take $\varepsilon = 1$. For every $\delta > 0$, choose $n \in \mathbb{N}$ such that $1/n < \delta$ by Archimedean Property and take $x = 1/n$ and $u = 1/(n + 1)$. Then $|x - y| < \delta$ and

$$|f(x) - f(u)| = |n - (n + 1)| = 1.$$

(d) It is uniformly continuous on $\mathbb{R}$. Note that for any $x, u \in \mathbb{R}$,

$$|f(x) - f(u)| = \left| \frac{1}{1 + x^2} - \frac{1}{1 + u^2} \right| = \frac{|x + u|}{(1 + x^2)(1 + u^2)} |x - u|.$$

Hence we estimate:

$$\frac{|x + u|}{(1 + x^2)(1 + u^2)} \leq \frac{|x| + |u|}{(1 + x^2)(1 + u^2)} \leq \frac{|x|}{1 + x^2} + \frac{|u|}{1 + u^2} \leq \frac{1}{2} + \frac{1}{2} = 1.$$

Then for any $\varepsilon > 0$, we can take $\delta = \varepsilon$. Then whenever $x, u \in \mathbb{R}$ and $|x - u| < \delta$,

$$|f(x) - f(u)| \leq |x - u| < \delta = \varepsilon.$$

**Remark.** Compare the proof of (c) to the paragraph under 5.4.2 in the textbook.

The following theorems is very helpful to establish the uniform continuity of functions. Some conditions on the domain of definition are required in addition to the continuity.

**Uniform Continuity Theorem** (c.f. 5.4.3)**.** *Let $I$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on $I$. Then $f$ is uniformly continuous on $I$.*

**Continuous Extension Theorem** (c.f. 5.4.8)**.** *A function $f$ is uniformly continuous on the interval $(a, b)$ if and only if it can be defined at the end-points $a$ and $b$ such that the extended function is continuous on $[a, b]$.*

# Lipschitz Functions

The following class of functions, called **Lipschitz functions**, are uniformly continuous. Thus, it gives another way for us to establish uniform continuity.

**Definition** (c.f. Definition 5.4.4)**.** Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. If there exists a constant $K > 0$ such that

$$|f(x) - f(u)| \leq K|x - u|, \quad \forall x, u \in A,$$

then $f$ is said to be a **Lipschitz function** or said to **satisfy a Lipschitz condition** on $A$.

**Theorem.** *A Lipschitz function on $A$ is uniformly continuous on $A$.*

**Example.** The function $f(x) = \sin x$ is a Lipschitz function on $\mathbb{R}$, and hence uniformly continuous on $\mathbb{R}$.

*Proof.* We will use the fact that $|\sin x| \leq |x|$ for any real number $x$. Note that

$$|\sin x - \sin u| = \left|2\cos\left(\frac{x+u}{2}\right)\sin\left(\frac{x-u}{2}\right)\right| \leq 2 \cdot 1 \cdot \left|\frac{x-u}{2}\right| = |x - u|.$$

Hence $f$ satisfies a Lipschitz condition on $\mathbb{R}$ with $K = 1$. □

**Exercise 2** (c.f. Section 5.4, Ex.6)**.** Show that if $f$ and $g$ are uniformly continuous on $A \subseteq \mathbb{R}$ and they are both bounded on $A$, then their product $fg$ is uniformly continuous on $A$. Can the assumption that both $f$ and $g$ are bounded be dropped?

**Solution.** Let $M_1$ and $M_2$ be the bound of $f$ and $g$ respectively. i.e.,

$$|f(x)| \leq M_1 \quad \text{and} \quad |g(x)| \leq M_2, \quad \forall x \in A.$$

Note that for any $x, u \in A$,

$$\begin{aligned} |fg(x) - fg(u)| &= |f(x)g(x) - f(u)g(u)| \\ &= |f(x)g(x) - f(x)g(u) + f(x)g(u) - f(u)g(u)| \\ &\leq |f(x)||g(x) - g(u)| + |g(u)||f(x) - f(u)| \\ &\leq M_1|g(x) - g(u)| + M_2|f(x) - f(u)| \end{aligned}$$

Let $\varepsilon > 0$. Since $f$ and $g$ are uniformly continuous on $A$, there exists $\delta_1 > 0$ such that

$$|f(x) - f(u)| < \frac{\varepsilon}{2M_2}, \quad \text{whenever } x, u \in A \text{ and } |x - u| < \delta_1.$$

Also, there exists $\delta_2 > 0$ such that

$$|g(x) - g(u)| < \frac{\varepsilon}{2M_1}, \quad \text{whenever } x, u \in A \text{ and } |x - u| < \delta_2.$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Then whenever $x, u \in A$ and $|x - u| < \delta$,

$$|fg(x) - fg(u)| \leq M_1|g(x) - g(u)| + M_2|f(x) - f(u)| < M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} = \varepsilon.$$

**Exercise 3** (c.f. Section 5.4, Ex.12)**.** Show that if $f$ is continuous on $[0, \infty)$ and uniformly continuous on $[a, \infty)$, for some positive constant $a$, then $f$ is uniformly continuous on $[0, \infty)$.

**Solution.** Let $\varepsilon > 0$. Since $f$ is uniformly continuous on $[a, \infty)$, there exists $\delta_1 > 0$ such that whenever $x, u \in [a, \infty)$ and $|x - u| < \delta_1$,

$$|f(x) - f(u)| < \varepsilon.$$

Then, by **Uniform Continuity Theorem**, $f$ is uniformly continuous on $[0, a + \delta_1]$. Hence there exists $\delta_2 > 0$ such that whenever $x, u \in [0, a + \delta_1]$ and $|x - u| < \delta_2$,

$$|f(x) - f(u)| < \varepsilon.$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Then whenever $x, u \in [0, \infty)$ and $|x - u| < \delta$, we have either $x, u \in [0, a + \delta_1]$ or $x, u \in [a, \infty)$. Hence in both cases, we have

$$|f(x) - f(u)| < \varepsilon.$$

Thus $f$ is uniformly continuous on $[0, \infty)$.

**Exercise 4** (c.f. Section 5.4, Ex.11)**.** If $g(x) = \sqrt{x}$ for $x \in [0, 1]$, show that there does not exist a constant $K$ such that $|g(x)| \leq K|x|$ for all $x \in [0, 1]$. This shows that $g$ is not a Lipschitz function on $[0, 1]$, but it is uniformly continuous on $[0, 1]$.

**Solution.** Suppose on a contrary that there exists some constant $K$ such that

$$|\sqrt{x}| \leq K|x|, \quad \forall x \in (0, 1].$$

Note that we must have $K \geq 1$ if we put $x = 1$. Now put $x = 1/4K^2 \in [0, 1]$, we have

$$\frac{1}{2K} \leq K \cdot \frac{1}{4K^2}.$$

It follows that $\frac{1}{2} \leq \frac{1}{4}$. This is a contradiction, hence there does not exist such constant $K$.