MATH2060B: Analysis II: Revision Exercise (2020)

1. Let f be a differentiable function on (a, b). Show the followings:

   (i) If f is unbounded, then so is f'. Does the converse hold?
   Proof: Assume that f' is bounded. Fix $c \in (a, b)$. The Mean Valued Theorem implies that for any $x \in (a, b)$, there exists an element ξ between x and c such that $f(x) = f(c) + f^{'}(\xi)(x - c)$. This shows that $f(x)$ is bounded if f' is bounded.
   The converse does not hold.
   Example, consider $f(x) = \sin \dfrac{1}{x}$ for $x \in (0, 1)$.

   (ii) If f' is bounded, then f^2 is uniformly continuous on (a, b).
   Proof: As f' is bounded, part (i) implies that f is bounded. Then the assertion is obtained by the Mean Value Theorem and the following equality immediately

   $$|f^2(x) - f^2(y)| = |f(x) - f(y)||f(x) + f(y)|$$

   for all $x, y \in (a, b)$.

2. Let $f(x) := sgn(\sin \frac{1}{x})$ for $x \in [-1, 1] \setminus \{0\}$ and $f(0) = 0$. Is f a Riemann integrable function?
   Proof: Notice that f is discontinuous at the points $\pm\frac{1}{n\pi}$ and $x = 0$. The result can shown by the following statement at once:
   "If a function $F : [-1, 1] \to \mathbb{R}$ is only discontinuous on the set $\{\pm\frac{1}{n\pi}\} \cup \{0\}$, then F is integrable on $[-1, 1]$."

3. Let $f \in R[a, b]$. Show the followings.

   (i) If f is continuous at some point $c \in [a, b]$ and $\int_a^b f^2 = 0$, then $f(c) = 0$.

   (ii) For all $\varepsilon > 0$ there is a subinterval $[c, d]$ of $[a, b]$ such that the oscillation $\omega[c, d] < \varepsilon$ of f over $[c, d]$, where $\omega[c, d] := \{|f(x') - f(x'')| : x', x'' \in [c, d]\}$.
   Proof (ii): Suppose that the part (ii) is not true. Hence, there is $\varepsilon > 0$ such that $\omega[c, d] \geq \varepsilon$ for any subinterval $[c, d]$. One the other hand, since $f \in R[a, b]$, there is a partition P on $[a, b]$ such that $\sum \omega_i(f, P)\triangle x_i < \frac{b-a}{2}\varepsilon$. Then by the hypothesis, we see that

   $$0 < \varepsilon(b - a) \leq \sum \omega_i(f, P)\triangle x_i < \frac{b - a}{2}\varepsilon.$$

   It leads to a contradiction.

4. Find $\lim_{n\to\infty}\int_0^1 \frac{x^n}{1+x}dx$.
Proof: Let $\varepsilon > 0$. Notice that for all $a \in (0,10$, we have $\frac{x^n}{1+x} \le \frac{1}{1+x}$ for all $x \in [0,a]$ and for all $n \in \mathbb{N}$. Therefore, for all $\varepsilon > 0$, we can choose $a \in (0,1)$ such that $\int_a^1 \frac{x^n}{1+x}dx \le (1-a) < \varepsilon$ for all n. On the other hand, we have $\int_0^a \frac{x^n}{1+x}dx \le a^n \ln(1+a)$ for all n. Thus, there is $N \in \mathbb{N}$ such that

$$\int_0^1 \frac{x^n}{1+x}dx \le (\int_0^a + \int_a^1)\frac{x^n}{1+x}dx < 2\varepsilon$$

for all $n \ge N$.

5. Let $f_n(x) = \frac{x}{1+n^2x^2}$ for $x \in \mathbb{R}$. Do the sequences (f_n) and $(f_n^{'})$ converge uniformly on $\mathbb{R}$?
Claim 1: (f_n) converges uniformly on $\mathbb{R}$. Notice that $|f_n(x)| \le \frac{1}{2n}$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Therefore the sequence (f_n) converges uniformly to 0 on $\mathbb{R}$.
Claim 2: $(f_n^{'})$ does not converge uniformly on $\mathbb{R}$. In fact, we see that $f_n^{'}(x) = \frac{1-n^2x^2}{1+n^2x^2}$. On the other hand, we have $\lim f_n^{'}(x) = 0$ for $x \ne 0$ and $\lim f_n^{'}(0) = 1$. Thus the pointwise limit function of $f_n^{'}$ is discontinuous at $x \ne 0$. Therefore, $(f_n^{'})$ does not converge uniformly on $\mathbb{R}$ since $f_n^{'}$ is continuous on $\mathbb{R}$ for each n.

6. Let $f_n(x) = \sum_{k=1}^{n}(-1)^{k+1}\frac{\sin kx}{k}$. Does (f_n) converge uniformly on $\mathbb{R}$? (Remark: this example was due to Cauchy which was restated in Abel's note).

7. Determine the following series whether converges uniformly on its domain.

   (i) $\sum_{n=1}^{\infty}(-1)^n(1-x)x^n$, for $x \in [0,1]$.
   Proof: Yes. Reason: Note that the convergence radius of the series $\sum(-1)^n x^n$ is 1 and is convergent at $x = 1$. Then by the Abel Theorem (see the note), the series $\sum(-1)^n x^n$ is uniformly convergent on $[0,1]$. Thus, the series $\sum_{n=1}^{\infty}(-1)^n(1-x)x^n = (1-x)\sum_{n=1}^{\infty}(-1)^n x^n$ converges uniformly converges on $[0,1]$.

   (ii) $\sum_{n=1}^{\infty}(1-x)x^n$, for $x \in [0,1]$.
   Proof: No. Reason: Let $s_n(x)$ be the n-th partial sum of the series. Then for each positive integer N, we see that $s_{2N+3}(x) - s_N(x) = x^{N+1} - x^{2N+2}$. In this case, we let $x_N = (1/2)^{1/(N+1)} \in [0,1]$. Then $s_{2N+3}(x_N) - s_N(x_N) = 1/4$. Then by the Cauchy Theorem, the series does not converges uniformly on $[0,1]$.

   (iii) $\sum_{n=1}^{\infty}\frac{x}{1+n^4x^2}$, for $x \in [0,\infty)$.
   Proof Yes. Note that $0 \le \frac{x}{1+n^4x^2} \le 1/(2n^2)$ for all $x \ge 0$ and for all n. Then the result is obtained by the M-test.

8. Find the convergence domain of the series $\sum_{n=1}^{\infty} \frac{(-)^n}{2n-1} (\frac{1-x}{1+x})^n$.

Proof: Answer: The convergence domain is $[0, \infty)$. Reason: Put $y := \frac{(-)^n}{2n-1} (\frac{1-x}{1+x})^n$. Note that the convergence domain of the power series of $\sum \frac{(-)^n}{2n-1} y^n$ is $(-1, 1]$. Note that $-1 < y \leq 1$ if and only if $x \geq 1$. The proof is finished.