THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2020)
Suggested Solution of Homework 7: Section 8.1: 5, 15, 22, 23

5. Evaluate $\lim((\sin nx)/(1+nx))$ for $x \in \mathbb{R}$, $x \geq 0$. (2 marks)

   Solution. If $x = 0$, then clearly, $\lim((\sin nx)/(1+nx)) = 0$. If $x > 0$ is fixed, then

$$\left|\frac{\sin nx}{1+nx}\right| \leq \frac{1}{1+nx} \leq \frac{1}{nx} \to 0 \quad \text{as } n \to \infty.$$

   Therefore, for each fixed $x \geq 0$, we conclude that $\lim((\sin nx)/(1+nx)) = 0$.

15. Show that if $a > 0$, then the convergence of the sequence in Exercise 5 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$. (3 marks)

   Solution. First, we show that for each $a > 0$, the sequence $\{f_n(x)\}$ of functions, defined by

$$f_n(x) = \frac{\sin nx}{1+nx} \qquad \text{for } x \geq 0,$$

   converges uniformly on the interval $[a, \infty)$. Since the pointwise limit is the zero function, we only need to check that this sequence converges uniformly to the zero function on $[a, \infty)$.

   Note that for every $x \in [a, \infty)$

$$\left|\frac{\sin nx}{1+nx}\right| \leq \frac{1}{1+nx} \leq \frac{1}{nx} \leq \frac{1}{na}$$

   Since for every $x \in [a, \infty)$, $|f_n(x)|$ shares the same bound $\frac{1}{na}$, and $\frac{1}{na} \to 0$ as $n \to \infty$, we have verified that the convergence on $[a, \infty)$ is uniform.

   However, the sequence of functions does not converge uniformly on $[0, \infty)$. It suffices to propose an $\epsilon > 0$ so that for any $N \in \mathbb{N}$, we can find some integer $n \geq N$ and some number $x_n \in [0, \infty)$ satisfying the inequality $|f_n(x_n) - 0| \geq \epsilon$.

   We propose $\epsilon = 1/3$. For each $N \in \mathbb{N}$, we consider $n = N$ and $x_n = \pi/(2n) \in [0, \infty)$. Hence,

$$|f_n(x_n) - 0| = \frac{1}{1+\frac{\pi}{2}} \geq \frac{1}{3} = \epsilon.$$

   This shows the claim.

22. Show that if $f_n(x) := x + 1/n$ and $f(x) := x$ for $x \in \mathbb{R}$, then (f_n) converges uniformly on $\mathbb{R}$ to f, but the sequence (f_n^2) does not converge uniformly on $\mathbb{R}$. (Thus the product of uniformly convergent sequences of functions may not converge uniformly.) (2 marks)

Solution. Notice that for every $x \in \mathbb{R}$,

$$|f_n(x) - f(x)| = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

Therefore, (f_n) converges uniformly on $\mathbb{R}$.

To see that (f_n^2) does not converge uniformly, we first observe that the pointwise limit of this sequence of functions is f^2. Therefore, it suffices to propose an $\epsilon > 0$ so that for any $N \in \mathbb{N}$, we can find some integer $n \geq N$ and some number $x_n \in \mathbb{R}$ satisfying the inequality $|f_n(x_n)^2 - f(x_n)^2| \geq \epsilon$.

We propose $\epsilon = 1$. For each $N \in \mathbb{N}$, we consider $n = N$ and $x_n = n \in \mathbb{R}$. Note that

$$|f_n(x_n)^2 - f(x_n)^2| = 2 + \frac{1}{n^2} > 1 = \epsilon$$

This shows the claim.

23. Let $(f_n), (g_n)$ be sequences of bounded functions on A that converge uniformly on A to f, g, respectively. Show that $(f_n g_n)$ converges uniformly on A to fg. (3 marks)

Solution. First, we will show that there is some $M > 0$ such that $|f_n(x)| \leq M$ for every $n \in \mathbb{N}$ and $x \in A$. Since the sequence (f_n) converges uniformly on A, by Cauchy criterion, there is some $N \in \mathbb{N}$, so that

$$\sup_{x \in A} |f_n(x) - f_m(x)| < 1 \quad \text{for any } n, m \geq N.$$

In particular,

$$|f_n(x) - f_m(x)| < 1 \quad \text{for any } n, m \geq N, \, x \in A.$$

We fix $m = N$. The assumption that f_N is a bounded function tells us that $\|f_N\| := \sup\{|f_N(x)| : x \in A\}$ exists. Therefore, we may conclude that $|f_n(x)| < 1 + \|f_N\|$ for every $n \geq N$ and $x \in A$, using triangle inequality. We can simply put $M = \max\{1 + \|f_N\|, \|f_1\|, \cdots, \|f_{N-1}\|\}$.

Without loss of generality, we can assume that $|f_n(x)|, |g_n(x)| \leq M$ for every $n \in \mathbb{N}$ and $x \in A$. Up to now, we have argued that the sequences (f_n), (g_n) of functions are uniformly bounded, independent of n.

By definition of uniform convergence, there is some $N \in \mathbb{N}$ such that $|f_n(x) - f(x)|, |g_n(x) - g(x)| < \epsilon/(2M)$ for all $n \geq N$ and $x \in A$. Now, for any $n \geq N$ and $x \in A$, we have

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x) - f(x)|\,|g_n(x)| + |g_n(x) - g(x)|\,|f(x)| \\ &< \frac{\epsilon}{2M}\, M + \frac{\epsilon}{2M}\, M = \epsilon \end{aligned}$$

This proves the claim. Remark: A sequence (f_n) of functions converges uniformly to f on the set A, is equivalent to say that $\lim\limits_{n \to \infty} \sup\limits_{x \in A} |f_n(x) - f(x)| = 0$.