THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2020)
Suggested Solution of Homework 3: Section 6.1: 4, 8, 9

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^2$ for x rational, $f(x) := 0$ for x irrational. Show that f is differentiable at $x = 0$, and find $f'(0)$. (2 marks)

   Solution. We claim that $f'(0) = 0$. Let $\epsilon > 0$. We choose $\delta = \epsilon$. For $0 < |x-0| < \delta$,

   (i) if x is rational, then $|\dfrac{f(x) - f(0)}{x - 0} - 0| = |\dfrac{x^2}{x} - 0| = |x| < \delta = \epsilon$;

   (ii) if x is irrational, then $|\dfrac{f(x) - f(0)}{x - 0} - 0| = 0 < \epsilon$.

   Therefore, $\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = 0$.

8. Determine where each of the following functions from $\mathbb{R}$ to $\mathbb{R}$ is differentiable and find the derivative: (1.5 marks each)

   (a) $f(x) := |x| + |x + 1|$,
   (b) $g(x) := 2x + |x|$,
   (c) $h(x) := x|x|$,
   (d) $k(x) := |\sin x|$.

   Solution.

   In the following, we use the fact that the function $|x|$ is differentiable on $\mathbb{R} \setminus \{0\}$ with derivative $\frac{x}{|x|}$, but it is not differentiable at 0.

   (a) By chain rule, we see that f is differentiable on $\mathbb{R} \setminus \{0, -1\}$, and the derivative is
   $$f'(x) = \frac{x}{|x|} + \frac{x + 1}{|x + 1|}.$$
   Moreover, f is not differentiable at either $x = 0$ or $x = -1$.

   (b) Clearly, g is differentiable on $\mathbb{R} \setminus \{0\}$, and the derivative is
   $$g'(x) = 2 + \frac{x}{|x|}.$$
   Moreover, g is not differentiable at the point $x = 0$.

   (c) By product rule, h is differentiable on $\mathbb{R} \setminus \{0\}$, and the derivative is
   $$h'(x) = |x| + x\frac{x}{|x|} = 2|x|.$$
   We claim that h is also differentiable at $x = 0$. Note
   $$\lim_{x\to 0} \frac{h(x) - h(0)}{x - 0} = \lim_{x\to 0} |x| = 0.$$

(d) By chain rule, k is differentiable at x whenever $\sin x \neq 0$. That is the set $\mathbb{R} \setminus (\pi\mathbb{Z})$. Moreover, the derivative is

$$k'(x) = \frac{\sin x}{|\sin x|} \cos x.$$

For $n \in \mathbb{Z}$,

$$\lim_{h\to 0} \frac{k(n\pi + h) - k(n\pi)}{h} = \lim_{h\to 0} \frac{|\sin h|}{h}.$$

The left hand limit is -1 while the right hand limit is 1, therefore the limit does not exist. We conclude that k is not differentiable at every point in $\pi\mathbb{Z}$.

9. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an **even function** [that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$] and has a derivative at every point, then the derivative f' is an **odd function** [that is, $f'(-x) = -f'(x)$ for all $x \in \mathbb{R}$]. Also prove that if $g : \mathbb{R} \to \mathbb{R}$ is a differentiable odd function, then g' is an even function. (2 marks)

Solution.

Using the formula $f(x) = f(-x)$, by chain rule, we have $f'(x) = f'(-x)\,(-1) = -f'(-x)$ whenever f is differentiable at $-x$. Hence, $f'(-x) = -f'(x)$ for all $x \in \mathbb{R}$.

By $g(x) = -g(-x)$, we have $g'(x) = -g'(-x)\,(-1) = g'(-x)$. Therefore, g' is an even function.