Homework 5 solutions

April 27, 2020

Problem 1. (a) Choose $\delta > 0$ small enough such that $\delta < r := \rho(p,q)$ and $\exp_p$ is a diffeomorphism from $B_{2\delta}(0) \subset T_pM$ onto its image. Let $S = \{z \in M : \rho(p,z) = \delta\}$. Since $S \subset M$ is compact, there exists $m \in S$ such that $\rho(m,q) = \inf_{z\in S} \rho(z,q)$. By triangle inequality, we have

$$\rho(p,m) + \rho(m,q) \geq \rho(p,q) \geq \delta + \inf_{z\in S} \rho(z,q) = \rho(p,m) + \rho(m,q).$$

So all the inequalities are in fact equalities. Since $m \in S$, there exists some $v \in T_pM$ with $\|v\| = 1$ such that $m = \exp_p(\delta v)$. Consider the geodesic $\gamma(t) = \exp_p(tv)$ and define

$$\tau := \sup\{t \in [0,r] : t + \rho(\gamma(t), q) = r\}.$$

Note that $\tau \geq \delta$. We claim that $\tau = r$, which then implies $\gamma(\tau) = q$ and thus $\gamma|_{[0,r]}$ is a geodesic of length r joining p to q. Suppose not, then $\tau < r$. Note that $\rho(\gamma(\tau), q) = r - \tau$. Choose $\epsilon > 0$ such that $\epsilon < r - \tau$ and $\exp_{\gamma(\tau)}$ is a diffeomorphism on $B_{2\epsilon}(0) \subset T_{\gamma(\tau)}M$ onto its image. By the same argument as above, we can extend the geodesic past $\gamma(\tau)$ such that it remains length minimizing. This contradicts that τ is the supremum. Finally, notice that this implies every closed and bounded subset (w.r.t. the metric ρ) of M is compact, hence (M, ρ) is a complete metric space.

(b) Let $\gamma : [0,T) \to M$ be a maximal geodesic with $\gamma(0) = p$ and $\gamma'(0) = v \in T_pM$ such that $T < \infty$ and γ cannot be defined at $t = T$. Choose a sequence $t_k \nearrow T$. The points $\{\gamma(t_k)\}$ forms a Cauchy sequence in (M, ρ) as

$$\sup_{\ell \geq k+1} \rho(\gamma(t_k), \gamma(t_\ell)) \leq |v| \sup_{\ell \geq k+1} |t_k - t_\ell| \to 0 \quad \text{ as } k \to \infty.$$

By completeness of (M, ρ), $\gamma(t_k)$ converges to some $q \in M$. Note that $|\gamma'(t_k)| = |v|$ for all k. After passing to a subsequence, the pair $(\gamma(t_k), \gamma'(t_k))$ converges to (q, w) for some $w \in T_qM$. Let $\sigma : [0, \epsilon] \to M$ be the geodesic with $\sigma(0) = q$, $\sigma'(0) = w$. For any sufficiently large k, the geodesics starting from $\gamma(t_k)$ with initial velocity $\gamma'(t_k)$ will be defined on the interval $[0, \epsilon/2]$, contradicting the maximality of T.

Problem 2. (a) The calculation is very similar to the first variation of length functional.

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\Big|_{s=0} E(\gamma_s) &= \int_0^1 \langle \gamma'(t), D_{\frac{\partial}{\partial t}} V(t) \rangle \, dt \\
&= \int_0^1 \frac{d}{dt} \langle \gamma'(t), V(t) \rangle - \langle D_{\frac{\partial}{\partial t}} \gamma'(t), V(t) \rangle \, dt \\
&= \langle \gamma'(t), V(t) \rangle|_{t=0}^{t=1} - \int_0^1 \langle D_{\frac{\partial}{\partial t}} \gamma'(t), V(t) \rangle \, dt.
\end{aligned}$$

(b) Since $D_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} = 0$ for all s, we have $D_{\frac{\partial}{\partial s}} D_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} = 0$. Hence

$$D_{\frac{\partial}{\partial t}} D_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s} = \sum_{k=1}^m R(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t}, e_k) e_k$$

where $\{e_k\}$ is any local orthonormal basis for TM. We get the desired conclusion by setting $s = 0$.

(c) Let $V(t)$ be a Jacobi field along a geodesic $\gamma(t) : [0,1] \to M$. Choose a path $\alpha : (-\epsilon, \epsilon) \to M$ such that $\alpha(0) = \gamma(0)$ and $\alpha'(0) = V(0)$. Let $X(s)$ be a vector field along α such that $X(0) = \gamma'(0)$ and $D_{\frac{\partial}{\partial s}} X(s)\Big|_{s=0} = D_{\frac{\partial}{\partial t}} V(t)\Big|_{t=0}$. Define $\gamma(t,s) := \exp_{\alpha(s)}(tX(s))$. Note that $\gamma(t,0) = \gamma(t)$ and one can easily check that $V(t) = \frac{\partial \gamma}{\partial s}\Big|_{s=0}$ by uniqueness of ODE solution.

Problem 3. Let $\{e_1, \cdots, e_m\}$ be a local orthonormal basis for $T\Sigma$. Then

$$\mathrm{div}_\Sigma N = \sum_{i=1}^{m} \langle D_{e_i} N, e_i \rangle = -\sum_{i=1}^{m} \langle N, D_{e_i} e_i \rangle = \langle N, -\sum_{i=1}^{m} (D_{e_i} e_i)^\perp \rangle = H.$$

Problem 4. (a) One can check directly that $\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \rangle = \langle \frac{\partial F}{\partial v}, \frac{\partial F}{\partial v} \rangle = 1$ and $\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial v} \rangle = 0$. (b) Clearly, $|F|^2 = 2$. The minimality can be checked by direct calculation of H.

Problem 5. Let $\pi : \tilde{M} \to M$ be the universal cover of M. Fix a point $\tilde{p} \in \tilde{M}$ such that $\pi(\tilde{p}) = p$. Lift α in M to a geodesic $\tilde{\alpha}$ in $\tilde{M}$, starting at $\tilde{p}$ and ending at some $\tilde{q} \in \tilde{M}$ with $\pi(\tilde{q}) = q$. Similarly, we can lift β to a geodesic $\tilde{\beta}$ on $\tilde{M}$ joining $\tilde{p}$ to some $\tilde{q}'$ with $\pi(\tilde{q}') = q$. Since α and β are homotopic with endpoints fixed, we can lift the homotopy to $\tilde{M}$ to conclude that $\tilde{\alpha}$ and $\tilde{\beta}$ are also homotopic with endpoints fixed. In particular, this shows that $\tilde{q} = \tilde{q}'$. So, we have two geodesics $\tilde{\alpha}$ and $\tilde{\beta}$ in $\tilde{M}$ joining $\tilde{p}$ to $\tilde{q}$. However, as M has non-positive sectional curvature, so does $\tilde{M}$. For any simply connected complete manifold with non-positive sectional curavture, any two points are joined by a unique geodesic. This show that $\tilde{\alpha}$ and $\tilde{\beta}$, hence α and β, are identical up to reparametrization.