Homework 3 solutions

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Problem 1. The pullback bundle $F^*E := \{(p, v) : p \in M,\ v \in E_{F(p)}\}$ is a vector bundle over M with projection onto the first factor. The pullback connection F^*D on F^*E can be locally defined as follows. Let $p \in M$, and $s_1, \cdots, s_r$ be a local basis of sections of E near $F(p) \in N$. Any local section s^* of F^*E near p can be expressed in the form $s^*(x) = \sum_{i=1}^r a^i(x) s_i(F(x))$. If X is a local vector field of M near p, then $(F^*D)_X(s^*)(x) = \sum_{i=1}^r X(a^i)(x)s_i(F(x)) + a^i(x)D_{F_*X}(s_i)(F(x))$. It is then straightforward to check that F^*D is well-defined independent of the choice of local trivialization given by $s_1, \cdots, s_r$. Note that the local trivialization $s_1, \cdots, s_r$ of E gives a local trivialization $s_1 \circ F, \cdots, s_r \circ F$ of F^*E. Under these trivializations, the curvature 2-forms of these connections are related by $F^*\Omega = \Omega^*$.

Problem 2. (a) Fix a local trivialization $s_1, \cdots, s_r$ of E with connection 1-forms $\{\omega^i_j\}$. For any $s = \sum_{i=1}^r a^i s_i \in \Gamma(E)$, we have $Ds = \sum_{i=1}^r da^i \otimes s_i + \sum_{i,j=1}^r a^j \omega^i_j \otimes s_i$. Differentiate once again, we obtain $D^2 s = -\sum_i^r da^i \wedge Ds_i + \sum_{i,j} d(a^j\omega^i_j) \otimes s_i - a^j\omega^i_j \wedge Ds_i = \sum_{i,j}(d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j)a^j s_i = \sum_{i,j} \Omega^i_j a^j s_i$. This shows $D^2 s = \Omega s$. (b) follows from (a) and induction on k.

Problem 3. (a) It can be checked directly that the induced connection satisfies all the conditions for a connection. The curvatures of E and E^* are the same in the following sense: if $s_1, \cdots, s_r$ is a local trivialization of E, then their duals $s_1^*, \cdots, s_r^*$ give a local trivialization of E^*, and the curvature 2-forms of E and E^* are related by $\Omega* = -\Omega^t$ (as endomorphism-valued 2-forms) under these local trivializations. (b) It is again straightforward to check that ∇ defines a connection. The curvatures are related by the formula $R^\nabla(X, Y)(s \otimes s') = (R^D(X, Y)s) \otimes s' + s \otimes (R^{D'}(X, Y)s')$.

Problem 4. For simplicity, we just treat the case $\dim(M) = 2$. Let s be the local extension of $s_0 \in E_P$ as described in the question. By construction, we have $(D_{\partial/\partial x_1}s)(x_1, 0) = 0$ and $(D_{\partial/\partial x_2}s)(x_1, x_2) = 0$. Since the connection is flat, covariant derivatives w.r.t. $\partial/\partial x_1$ and $\partial/\partial x_2$ commutes. Therefore, we have $D_{\partial/\partial x_2}D_{\partial/\partial x_1}s = D_{\partial/\partial x_1}D_{\partial/\partial x_2}s = 0$ near the origin. Moreover, since $(D_{\partial/\partial x_1}s)(x_1, 0) = 0$, we obtain $(D_{\partial/\partial x_1}s)(x_1, x_2) = 0$ as well.

Problem 5. We only prove (b) here since it implies (a). Suppose γ_0 and γ_1 are two closed loops based at $p \in M$. If γ_0 and γ_1 are homotopic to each other, there exists a homotopy $\gamma = \gamma(s, t) : [0, 1] \times [0, 1] \to M$ such that $\gamma_0(t) = \gamma(0, t)$ and $\gamma_1(t) = \gamma(1, t)$. Divide the square $[0, 1] \times [0, 1]$ into small sub-squares. By Problem 4, the parallel transport along any such small sub-square is the identity map since the connection is flat. By propagating these loops using the homotopy, this implies that the parallel transports along γ_0 and γ_1 are the same. Hence, $\rho : \pi_1(M, P) \to GL(k, \mathbb{R})$ is well-defined and clearly a group homomorphism. (c) By the equivalence proved in lectures, it suffices to construct local basis of parallel sections of E over M. Let $p \in M$ and fix any $\tilde{p} \in \tilde{M}$ which projects to p under the covering $\tilde{M} \to M$. Since $\tilde{E}$ is trivial, we can take a local parallel basis of sections of $\tilde{E}$ near $\tilde{p}$. By restricting to a smaller neighborhood so that the covering $\tilde{M} \to M$ gives a local diffeomorphism. These parallel sections of $\tilde{E}$ near $\tilde{p}$ clearly descend to parallel sections in E near p. To see that E has holonomy ρ, take any non-trivial loop $\gamma \in \pi_1(M, p) \cong \Gamma$ which lifts to a path $\tilde{\gamma}$ in $\tilde{M}$ joining $\tilde{p}$ to $\gamma(\tilde{p})$. Since $\tilde{E}$ is globally trivial, the parallel transport along $\tilde{\gamma}$ is identity. Hence, under the action of Γ on $\tilde{E}$, the parallel transport along γ is given by $\rho(\gamma)$.