

# MATH 5061 Riemannian Geometry

## Solution to Problem Set 5

### Problem 1

(a). Let  $B_\delta(p)$  be a small ball centered at  $p$  such that for any  $s_1, s_2 \in \overline{B_\delta(p)}$ , there is a unique geodesic  $\gamma$  jointing  $s_1, s_2$  such that  $\rho(s_1, s_2) = \text{Length of } \gamma$ . We call this ball as the totally normal ball at  $p$ . Let  $S_\delta(p)$  be the boundary of  $B_\delta(p)$ . Note that  $S_\delta(p)$  is a compact set, we can find some  $s \in S_\delta(p)$  such that  $\rho(s, q)$  attains a minimum on  $S_\delta(p)$ . So we can find a minimizing geodesic  $\gamma(t)$  with  $\gamma(0) = p, \gamma(\delta) = s$  and  $\|\gamma'(0)\| = 1$ . By the definition of exponential map, we have  $\exp_p(\delta v) = s$  where  $v = \gamma'(0) \in T_p M$ . Let  $l = \rho(p, q)$ , we are going to show  $\exp_p(lv) = q$ . Since  $\exp_p(tv)$  defined for all  $t \in \mathbb{R}$ , we actually can extend the definition of  $\gamma(t)$  for  $t \in \mathbb{R}$  by  $\gamma(t) = \exp_p(tv)$ .

We consider the following equation.

$$\rho(\gamma(t), q) = l - t \tag{1}$$

Let  $A = \{t \in (0, l] : (??) \text{ holds for } t\}$ . Clearly  $A \neq \emptyset$  since  $\delta \in A$  (Triangle inequality  $\implies \rho(s, q) \geq l - \delta$ . If  $\rho(s, q) = l_0 > l - \delta$ , then any piecewise smooth curve jointing  $p, q$  will  $\geq l_0 + \delta$  since they will pass through  $S_\delta(p)$ ).

Note that  $A$  is closed in  $(0, l]$  by the continuous of distance. So let's show if  $t_0 \in A$  and  $t_0 \neq l$ , then we can find  $\delta' > 0$  such that  $t_0 + \delta' \in A$ . Still we choose a totally normal ball  $B_{\delta'}(\gamma(t_0))$  such that  $p, q \notin B_{\delta'}(\gamma(t_0))$ . So we know  $\delta' \leq \rho(\gamma(t_0), q) = l - t_0 \implies t_0 + \delta' \leq l$ . Again, we can find some  $s' \in S_{\delta'}(\gamma(t_0))$  such that  $\rho(s, q)$  attains a minimum on  $S_{\delta'}(\gamma(t_0))$ . We claim  $s' = \gamma(t_0 + \delta')$ . If not, we note  $\rho(s', \gamma(t_0 - \delta')) < \rho(s', \gamma(t_0)) + \rho(\gamma(t_0), \gamma(t_0 - \delta')) = 2\delta'$  by the definition of totally normal ball. Hence  $\rho(s', p) < t_0 + \delta'$ . Again by triangle inequality,  $\rho(q, s') \geq l - \rho(x', p) > l - t_0 - \delta'$ . Since any curves jointing  $\gamma(t_0), q$  will pass through  $S_{\delta'}(\gamma(t_0))$ , we actually know  $\rho(q, \gamma(t_0)) \geq \rho(x', q) + \delta' > l - t_0$ , a contradiction with  $t_0 \in A$ . So we should have  $s' = \gamma(t_0 + \delta')$ . Still by triangle inequality  $\rho(\gamma(t_0 + \delta'), p) \geq l - t_0 - \delta'$  but  $\rho(\gamma(t_0 + \delta'), p) > l - t_0 - \delta'$  cannot hold by the same reason. Hence  $\rho(\gamma(t_0 + \delta'), q) = l - t_0 - \delta' \implies t_0 + \delta' \in A$ .

The above steps show  $\sup A \in A$  by the closeness and moreover  $\sup A = l$ . Hence  $l \in A$  and  $\gamma(l) = q$ . The  $\gamma$  is the geodesic jointing  $p, q$  realized the distance  $\rho(p, q)$ .

To prove  $(M, \rho)$  is complete, note for any Cauchy sequence  $(p_i)$ , we know  $\rho(p_i, p)$  is bounded by Triangle inequality. Suppose  $\rho(p_i, p) < M$  for all  $i$ , we know  $p_i$  in the image of  $\overline{B_M(p)}$  under the map  $\exp_p$ . Note  $\overline{B_M(p)}$  is compact, so does the set  $\exp_p(\overline{B_M(p)})$ . Hence we can find a convergent subsequence of  $(p_i)$  and indeed the whole sequence will have the same limit since it is Cauchy.

(b). Let's suppose  $\exp_p$  is not defined on the whole  $T_pM$ . That means there is a geodesic  $\gamma(t)$  with  $\gamma(0) = p$  is not defined for some  $t$ . WLOG, we assume  $\|\gamma'(0)\| = 1$ . By the existence of geodesic, we know there is a largest open interval  $(-s_0, s_1)$  such that  $\gamma(t)$  is well-defined. Let  $t_i \in (-s_0, s_1)$  such that  $t_i \rightarrow s_1$ . Note  $\rho(\gamma(t_i), \gamma(t_j)) \leq |t_i - t_j|$ ,  $\gamma(t_i)$  is Cauchy and we can find  $q \in M$  such that  $\gamma(t_i) \rightarrow q$ .

Now let  $B_\delta(q)$  be a totally normal ball at  $q$ . We can find  $N$  large such that  $p_i \in B_{\frac{\delta}{2}}(q)$  and  $|t_i - s_1| < \frac{\delta}{2}$  for all  $i > N$ . Note that any two points in  $B_\delta(p)$  can be joined by a minimizing geodesic, we know the exponential map  $\exp_{p_i}$  defined for all  $\|v\| \leq \frac{\delta}{2}$ . Let's consider two points  $p_i, p_j$  with  $N < i < j$  and they're joined by a minimizing geodesic  $\gamma(t), t \in [t_i, t_j]$ . But note  $\exp_{t_j}(t\gamma'(t_j))$  exists for  $t \in [-\frac{\delta}{2}, \frac{\delta}{2}]$ , we know  $\gamma(t)$  is well-defined when  $t \in [t_j, t_j + \frac{\delta}{2}]$ . Note  $t_j + \frac{\delta}{2} > s_1$ , so it contradicts with the choice of  $\gamma$ . Hence  $\exp_p$  is defined on the whole  $T_pM$ .

## Problem 2

Suppose  $\gamma(t)$  defined on  $[0, T]$  with  $\|\gamma'(0)\| = 1$ . Let  $\tilde{\gamma}(s) : (-\varepsilon, \varepsilon) \rightarrow M$  be the geodesic starting from  $\gamma(0)$  with initial velocity  $V(0)$ . So we consider the variation of  $\gamma$  defined by

$$f(t, s) = \exp_{\tilde{\gamma}(s)}(tW(s))$$

where  $W(s)$  be the vector field along  $\tilde{\gamma}$  with  $W(0) = \gamma'(0)$  and  $\frac{DW}{ds}(0) = \frac{DV}{dt}(0)$ .

Clearly  $f(t, 0) = \exp_{\gamma(0)}(t\gamma'(0)) = \gamma(t)$ , so  $f$  is indeed a variation of  $\gamma$ .

Note that the variation of geodesic will give the Jacobi field. That is, if we define  $\tilde{V}(t) = \frac{\partial f}{\partial s}(t, 0)$ , the vector field along  $\gamma$ , then note  $\frac{D}{dt} \frac{\partial f}{\partial t} = 0$ , we have

$$\begin{aligned} 0 &= \frac{D}{ds} \frac{D}{dt} \frac{\partial f}{\partial t} = \frac{D}{dt} \frac{D}{ds} \frac{\partial f}{\partial t} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) \frac{\partial f}{\partial t} \\ &= \frac{D}{dt} \frac{D}{dt} \frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) \frac{\partial f}{\partial t} = \nabla_{\gamma'} \nabla_{\gamma'} \tilde{V} + R(\gamma', \tilde{V})\gamma' \end{aligned}$$

which shows  $\tilde{V}$  is indeed a Jacobi field.

Note that  $V(0) = \tilde{V}(0) = \frac{\partial f}{\partial s}(0, 0)$  and

$$\nabla_{\gamma'(0)} \tilde{V}(0) = \frac{D}{ds} \frac{\partial f}{\partial t}(0, 0) = \frac{D}{ds} W(0) = \frac{DV}{dt}(0) = \nabla_{\gamma'(0)} V(0)$$

Hence  $V = \tilde{V}$  along  $\gamma$  by the uniqueness of the ODE solutions.

So  $V$  arises as the variation of the geodesic.

## Problem 3

(a).

"  $\implies$  " Suppose  $X$  is a Killing vector field. Then

$$\frac{d}{dt} \Big|_{t=0} \varphi_t^* g = \mathcal{L}_X g = 0$$

where  $\varphi_t$  generated by  $X$  by the definition of Lie derivative of tensor. Hence

$$\frac{d}{dt} \Big|_{t=t_0} \varphi_t^* g = \frac{d}{dt} \Big|_{t=0} \varphi_{t_0}^* \circ \varphi_t^* g = \varphi_{t_0}^* \left( \frac{d}{dt} \Big|_{t=0} \varphi_t^* g \right) = \varphi_{t_0}^* (\mathcal{L}_X g) = 0$$

by the properties of flow. So  $\varphi_t^*g = g$ .

"  $\Leftarrow$  " Suppose  $\varphi_t^*g = g$  for all  $t$ , then

$$\mathcal{L}_X g = \frac{d}{dt}\Big|_{t=0} \varphi_t^*g = \frac{d}{dt}\Big|_{t=0} g = 0$$

(b). Let  $\varphi_t$  be the flow generated by  $X$ . So  $\varphi_t$  will be the isometries of  $M$ . Hence for any geodesic  $\gamma(s)$ , the variation of  $\gamma$  defined by  $\gamma_t(s) = \varphi_t(\gamma(s))$  is geodesic for every  $t \in \mathbb{R}$ . Hence the vector field  $V = \frac{\partial \gamma_t}{\partial t} = \frac{\partial}{\partial t} \varphi_t = X$  along  $\gamma(s)$  is a Jacobi field.

(c). Let  $A = \{p \in M : X, \nabla_Y X \text{ vanished at } p \text{ for all } Y(p) \in T_p M\}$ . Clearly  $A \neq \emptyset$  is closed. We show  $A$  is open, too. For any  $p \in M$ , we choose a small ball  $B_\delta(p)$  that for every point in  $q$ , there is a unique minimizing geodesic  $\gamma_{p,q}$  in  $B_\delta(p)$  jointing  $p, q$ . Note that  $X$  is a Jacobi field along  $\gamma_{p,q}$  that  $X, \nabla_{\gamma'_{p,q}(0)}$  vanish at  $p$ . But by the uniqueness of Jacobi field when given  $V(0), \nabla_{\gamma'(0)} V(0)$ , we know  $X$  should be the zero vector field. Hence  $X$  will be zero in the whole ball  $B_\delta(p)$ . So  $B_\delta(p) \subset A$ .

Since  $M$  is connect and  $A$  is open and closed at the same time, we know  $A = M$ .

#### Problem 4

Consider the projective space  $\mathbb{R}\mathbb{P}^n$  when  $n$  is odd. It is a quotient of  $\mathbb{S}^n$  under antipodal map  $\varphi$ . Since  $\varphi$  is orientation preserving when  $n$  is odd, we know  $\mathbb{R}\mathbb{P}^n$  is orientable and moreover  $\mathbb{R}\mathbb{P}^n$  has positive sectional curvature by the properties of covering map. Hence Synge theorem does not hold in odd dimensions.