THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 5 (February 24)

Monotone Convergence Theorem. *A monotone sequence of real numbers is convergent if and only if it is bounded. Furthermore,*

(a) If (x_n) is a bounded increasing sequence, then $\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$.

(b) If (y_n) is a bounded decreasing sequence, then $\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}$.

Example 1. Let $Z = (z_n)$ be the sequence of real numbers defined by

$$z_1 := 1, \quad z_{n+1} := \sqrt{2z_n} \quad \text{for } n \in \mathbb{N}.$$

Show that $\lim(z_n) = 2$.

Example 2 (*Euler number e*)**.** Let $e_n := (1 + 1/n)^n$ for $n \in \mathbb{N}$. Show that the sequence $E = (e_n)$ is bounded and increasing, hence convergent. The limit of this sequence is called the *Euler number*, and it is denoted by e.

Example 3. Establish the convergence and find the limits of the following sequences.

(a) $((1 + 1/n)^{n+1})$

(b) $\left(\left(1 + \frac{1}{n+1}\right)^n\right)$

(c) $((1 - 1/n)^n)$

Classwork

1. Let $y_1 := \sqrt{p}$, where $p > 0$, and $y_{n+1} := \sqrt{p + y_n}$ for $n \in \mathbb{N}$. Show that (y_n) converges and find the limit. (Hint: $1 + 2\sqrt{p}$ is one upper bound.)

2. Let $b_n = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!}$ for $n \in \mathbb{N}$. Show that (b_n) is convergent. Furthermore, show that
$$\lim(b_n) = \lim(e_n) = e.$$