MATH2050a Mathematical Analysis I

Exercise 2 suggested Solution

5. Use the definition of the limit of a sequence to establish the following limits.

(a) lim $\frac{n}{n^2+1}$=0 (b)lim$\frac{2n}{n+1}$=2

Solution:

(a) A sequence $\{x_n\}$ is said to converge to x, or that x is the limit of $\{x_n\}$, if for every $\epsilon > 0$, there exists a natural number n_ϵ, such that for all $n \geq n_\epsilon$, we have $|x_n - x| < \epsilon$.

since $|(\frac{n}{n^2+1}) - 0| < \frac{n}{n^2} = \frac{1}{n}$, following from Archimedean property, there exists $n_\epsilon > \frac{1}{\epsilon}$, so $\forall n > n_\epsilon$

$$|(\frac{n}{n^2+1}) - 0| < \frac{n}{n^2} = \frac{1}{n} < \epsilon$$

we have lim $\frac{n}{n^2+1}$=0

(b) since $|\frac{2n}{n+1} - 2| = |\frac{2n-2n-1}{n+1}| = \frac{1}{n+1}$, for each $\epsilon > 0$, similar with 5(a), there exists $(n_\epsilon + 1) > \frac{1}{\epsilon}$, so $\forall n > n_\epsilon$

$$|\frac{2n}{n+1} - 2| = |\frac{2n-2n-1}{n+1}| = \frac{1}{n+1} < \frac{1}{n_\epsilon+1} < \epsilon$$

Hence, we have lim$\frac{2n}{n+1}$=2.

14. Let $b \in R$ satisfy $0 < b < 1$. Show that lim(nb^n)=0.[Hint:Use the Binomial Theorem as the example 3.1.11(d)]

Solution:

Let $b \in R$ and $0 < b < 1$, we want to show lim(nb^n)=0. Since $0 < b < 1$, we obtain $\frac{1}{b} > 1$. Let $\frac{1}{b} = 1 + t$, where $t > 0$. Then we have

$$(nb^n) = \frac{n}{(1/b)^n} = \frac{n}{(1+t)^n}$$

By the Binomial Theorem , since $n \geq 1$, we have

$1 + nt + \frac{1}{2}n(n-1)t^2 + ... \geq 1 + \frac{1}{2}n(n-1)t^2 \geq \frac{1}{2}n(n-1)t^2$

It follows that $\frac{n}{(1+t)^n} \leq \frac{2}{(n-1)t^2}$, $\forall \epsilon > 0$, there exists $n_\epsilon \in N$, such that $n_\epsilon > \frac{2}{t^2\epsilon}$, hence, $\forall n - 1 > n_\epsilon$,

$$|nb^n - 0| = \frac{n}{(1+t)^n} \leq \frac{2}{(n-1)t^2} \leq \frac{2}{(n_\epsilon)t^2} \leq \epsilon$$

Hence,lim(nb^n)=0.

23. Show that if $\{x_n\}$ and $\{y_n\}$ are convergent sequences, then the sequence $\{u_n\}$ and $\{v_n\}$ defined by $u_n := max\{x_n, y_n\}$ and $v_n := min\{x_n, y_n\}$ are also convergent. (See Exercise 2.2.18.)

Solution:

according to Exercise 2.2.18, $u_n = \frac{1}{2}(x_n + y_n + |x_n - y_n|)$, and $v_n = \frac{1}{2}(x_n + y_n - |x_n - y_n|)$. Since $\{x_n\}$ and $\{y_n\}$ are convergent sequences,assuming that lim$x_n = a$, lim$y_n = b$. Therefore, $\forall \epsilon > 0$, there exist $N_1$and N_2, such that $\forall$n$\geq N_1, k \geq N_2$, we have

$$|x_n - a| < \epsilon \quad |y_k - b| < \epsilon$$

Let $N_3 \geq N_1 + N_2$, and so $\forall$n$\geq N_3$,$|x_n - a| < \epsilon$, $|y_n - b| < \epsilon$

So $\forall n \geq N_3$, $|x_n + y_n - (a+b)| < 2\epsilon$, and $||x_n - y_n| - |a-b|| < 2\epsilon$, which means that $|u_n - \frac{1}{2}(a+b+|a-b|)| < |x_n + y_n - (a+b)| + ||x_n - y_n| - |a-b|| < 4\epsilon$.

Hence, $\{u_n\}$ is a convergent sequence, and the limit point is $limx_n + limy_n - |limx_n - limy_n|$. Similarly, we can prove that $\{v_n\}$ converges.