Sequence of Functions

Definition (c.f. Definition 8.1.1)**.** Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ and let f be a function defined on A. (f_n) is said to *converge (pointwisely)* to f on A if $(f_n(x))$ converges to $f(x)$ for each $x \in A$. In this case, we denote

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{or} \quad f = \lim_{n\to\infty} f_n.$$

Definition (c.f. Definition 8.1.4)**.** Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ and let f be a function defined on A. (f_n) is said to *converge uniformly* to f on A if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N, \quad \forall x \in A.$$

Remark. Consider the ε-N notation for the two convergences. Note that N depends on **both** of ε and x for pointwise convergence. On the other hand, N depends **only** on ε for uniform convergence.

Example 1 (c.f. Section 8.1, Ex.8 & Ex.18)**.** Let $f_n : [0, \infty) \to \mathbb{R}$ be defined by

$$f_n(x) = xe^{-nx}.$$

Find the pointwise limit of (f_n) and show that the convergence on $[0, \infty)$ is uniform.

Solution. We first find the pointwise limit of f_n. If $x = 0$, then $f_n(x) = 0$ for all n. Hence

$$\lim_{n\to\infty} f_n(0) = 0.$$

If $x > 0$, note that $e^{-nx} \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} f_n(x) = x \cdot \lim_{n\to\infty} e^{-nx} = 0.$$

Now we show that (f_n) converges uniformly to the zero function. We need to find the maximum value of $f_n(x) = xe^{-nx}$ on $[0, \infty)$ for each n. Differentiating f_n gives

$$f_n'(x) = (1 - nx)e^{-nx}.$$

Hence it has only one critical point at $x = 1/n$. Now at the endpoints and critical point,

$$f_n(0) = 0, \quad f_n(1/n) = \frac{1}{ne} \quad \text{and} \quad \lim_{x\to\infty} f_n(x) = 0.$$

It follows that the maximum value of f_n is $1/ne$. Hence

$$|xe^{-nx} - 0| = xe^{-nx} \leq \frac{1}{ne}, \quad \forall n \in \mathbb{N}, \quad \forall x \geq 0.$$

Let $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that $1/N < e\varepsilon$. Then

$$|xe^{-nx} - 0| \leq \frac{1}{ne} \leq \frac{1}{Ne} < \varepsilon, \quad \forall n \geq N, \quad \forall x \geq 0.$$

It follows by definition that (f_n) converges to the zero function uniformly.

The following theorem is an analogue of the **Cauchy Criterion** for sequence of functions.

Theorem. *Let (f_n) be a sequence of real-valued functions defined on $A \subseteq \mathbb{R}$. Then (f_n) converges uniformly on A if and only if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$|f_n(x) - f_m(x)| < \varepsilon, \quad \forall m, n \geq N, \quad \forall x \in A.$$

Example 2 (c.f. Section 8.1, Ex.4 & Ex.14)**.** Let $f_n : [0, \infty) \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{x^n}{1 + x^n}.$$

Let $0 < b < 1$. Show that (f_n) converges uniformly on $[0, b]$ but not uniformly on $[0, 1]$.

Solution. The pointwise limit of (f_n) is given by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1 + x^n} = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1/2 & \text{if } x = 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Let's show that (f_n) converges uniformly to the zero function on $[0, b]$. Note that

$$\left| \frac{x^n}{1 + x^n} - 0 \right| = \frac{x^n}{1 + x^n} \leq \frac{b^n}{1 + 0^n} = b^n, \quad \forall n \in \mathbb{N}, \quad \forall x \in [0, b].$$

Let $\varepsilon > 0$. Since $0 < b < 1$, $b^n \to 0$ as $n \to \infty$. Hence we can choose $N \in \mathbb{N}$ such that $b^n < \varepsilon$ whenever $n \geq N$. It follows that

$$\left| \frac{x^n}{1 + x^n} - 0 \right| \leq b^n < \varepsilon, \quad \forall n \geq N, \quad \forall x \in [0, b].$$

To see that (f_n) does not converge uniformly on $[0, 1]$, we need to show that there exist $\varepsilon > 0$ such that whenever $N \in \mathbb{N}$, there exists $n \geq N$ and $x \in [0, 1]$ such that

$$|f_n(x) - f(x)| \geq \varepsilon.$$

For each $N \in \mathbb{N}$, take $n = N$ and $x = 2^{-1/n}$. Then $n \geq N$, $x \in [0, 1]$ and

$$|f_n(x) - f(x)| = \left| \frac{x^n}{1 + x^n} - 0 \right| = \frac{1/2}{1 + 1/2} = \frac{1}{3}.$$

This shows that the convergence is not uniform on $[0, 1]$.

Remark. Due to the proposition below, it suffices to show that (f_n) does not converge uniform to its pointwise limit f.

Proposition. *Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ and let f be a function defined on A. If (f_n) converges uniformly to f on A, then (f_n) converges to f on A.*

Observe that in **Example 2**, the argument for non-uniform convergence is similar to the lemma below.

Lemma (c.f. Lemma 8.1.5)**.** *Let (f_n) be a sequence of real-valued functions defined on $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. Then (f_n)* ***does not*** *converge uniformly to f on A if and only if there exists $\varepsilon > 0$, a subsequence (f_{n_k}) of (f_n) and a sequence (x_k) in A such that*

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon, \quad \forall k \in \mathbb{N}\,.$$

Proof. ($\Rightarrow$) Suppose f_n does not converge uniformly to f on A. By definition, there exists $\varepsilon > 0$ such that whenever $N \in \mathbb{N}$, there exists $n \geq N$ and $x \in A$ such that

$$|f_n(x) - f(x)| \geq \varepsilon.$$

We construct the sequences (f_{n_k}) and (x_k) as follows:

- For $k = 1$, consider $N = 1$. Then there exists $n_1 \geq 1$ and $x_1 \in A$ such that

$$|f_{n_1}(x_1) - f(x_1)| \geq \varepsilon.$$

- For $k = 2$, consider $N = n_1 + 1$. Then there exists $n_2 > n_1$ and $x_2 \in A$ such that

$$|f_{n_2}(x_2) - f(x_2)| \geq \varepsilon.$$

- Similarly for $k = 3, 4, ...$, consider $N = n_{k-1} + 1$. Then there exists $n_k > n_{k-1}$ and $x_k \in A$ such that

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon.$$

The implication is then clear from the construction. ($\Leftarrow$) We need to show that there exists $\varepsilon > 0$ such that whenever $N \in \mathbb{N}$, there exists $n \geq N$ and $x \in A$ such that

$$|f_n(x) - f(x)| \geq \varepsilon.$$

Obviously, we need to take $\varepsilon > 0$ as in the assumption. For each $N \in \mathbb{N}$, we can choose $k \in \mathbb{N}$ such that $n_k \geq N$. Taking $n = n_k$ and $x = x_k$, we have $n \geq N$, $x \in A$ and

$$|f_n(x) - f(x)| = |f_{n_k}(x_k) - f(x_k)| \geq \varepsilon.$$

The result follows. $\square$

Remark. In **Example 2**, $n_k = k$ and $x_k = 2^{-1/k}$.

Example 3 (c.f. Section 8.1, Ex.24)**.** Let (f_n) be a sequence of functions that converges uniformly to f on A and that satisfies $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in A$. If g is a continuous function defined on the interval $[-M, M]$, show that the sequence $(g \circ f_n)$ converges uniformly to a well-defined function $g \circ f$ on A.

Solution. To see that the function $g \circ f$ is well-defined, i.e., $f(x) \in [-M, M]$ for all $x \in A$, we can simply take limits on both sides of the inequality. We then need to show that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|g(f_n(x)) - g(f(x))| < \varepsilon, \quad \forall n \geq N, \quad \forall x \in A.$$

Let $\varepsilon > 0$. Since g is continuous on $[-M, M]$, g is uniformly continuous on $[-M, M]$. i.e., there exist $\delta > 0$ such that whenever $|u - v| < \delta$ and $u, v \in [-M, M]$,

$$|g(u) - g(v)| < \varepsilon.$$

Now since (f_n) converges uniformly to f, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \delta, \quad \forall n \geq N, \quad \forall x \in A.$$

Hence whenever $n \geq N$ and $x \in A$, take $u = f_n(x)$ and $v = f(x)$. Then we have $|u - v| < \delta$ and $u, v \in [-M, M]$. Therefore

$$|g(f_n(x)) - g(f(x))| < \varepsilon.$$