Substitution Theorem

Theorem (c.f. Theorem 2.24 of Lecture Note)**.** *Let $f \in \mathcal{R}[c,d]$ and let $\varphi : [a,b] \to [c,d]$ be a strictly increasing $\mathcal{C}^1$ function with $\varphi(a) = c$ and $\varphi(b) = d$. Then $f \circ \varphi \in \mathcal{R}[a,b]$ and*

$$\int_a^b f(\varphi(x))\varphi'(x)dx = \int_c^d f(x)dx.$$

Remark. The assumption that φ being strictly increasing is restrictive. It can be relaxed but the proof will become complicated. However, it is essential that φ is $\mathcal{C}^1$. i.e., φ is differentiable with a continuous derivative.

The Improper Integral

Definition. Let $-\infty < a < b \leq \infty$ and f be a function defined on $[a,b)$. Suppose $f \in \mathcal{R}[a,c]$ for all $c \in (a,b)$. The *improper integral* of f over $[a,b)$ is defined by

$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx,$$

provided that the limit exist. In this case, the improper integral is said to be *convergent*.

Remark. Similarly, we can define the improper integrals for functions defined on $(a,b]$, where $-\infty \leq a < b < \infty$.

Definition. Let $-\infty \leq a < c < b \leq \infty$ and f be a function defined on (a,b). The *improper integral* of f over (a,b) is defined by

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

provided that both improper integrals on the right-hand-side converges.

Theorem. *Let f be a function defined on $[a,\infty)$. Suppose $f \in \mathcal{R}[a,c]$ for all $c \in (a,\infty)$. The improper integral of f over $[a,\infty)$ is convergent if and only if for all $\varepsilon > 0$, there exists $M > a$ such that whenever $A_2 > A_1 > M$,*

$$\left| \int_{A_1}^{A_2} f(x)dx \right| < \varepsilon.$$

Theorem. *Let f be a function defined on $[a,b)$. Suppose $f \in \mathcal{R}[a,c]$ for all $c \in (a,b)$. The improper integral of f over $[a,b)$ is convergent if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $A_1, A_2 \in [a,b)$ and $b - \delta < A_1 < A_2 < b$,*

$$\left| \int_{A_1}^{A_2} f(x)dx \right| < \varepsilon.$$

Remark. We have similar characterisations for the convergence of improper intergrals over $(-\infty, b]$ and $(a,b]$.

Example 1 (2016-17 Final Q2)**.** Let $f : [1, \infty) \to \mathbb{R}$ be a function defined by

$$f(x) = \frac{\sin x}{x}.$$

(i) Show that the improper integral $\int_1^\infty f(x)dx$ is convergent.

(ii) Show that the improper integral $\int_1^\infty |f(x)|dx$ is divergent.

Solution. This shows that $f \in \mathcal{R}[a, b] \implies |f| \in \mathcal{R}[a, b]$ doesn't hold for improper integrals.

(i) Note that from the **Integration by Parts**,

$$\int_1^T f(x)dx = \int_1^T \frac{\sin x}{x}dx = \left[-\frac{\cos x}{x}\right]_1^T - \int_1^T \frac{\cos x}{x^2}dx, \quad \forall T > 1.$$

Hence it suffices to show that the improper integral $\int_1^\infty \frac{\cos x}{x^2}dx$ is convergent.

Note that whenever $A_2 > A_1 > 1$, we have

$$\left|\int_{A_1}^{A_2} \frac{\cos x}{x^2}dx\right| \le \int_{A_1}^{A_2} \frac{|\cos x|}{x^2}dx \le \int_{A_1}^{A_2} \frac{1}{x^2}dx = \frac{1}{A_1} - \frac{1}{A_2} \le \frac{1}{A_1}.$$

For any $\varepsilon > 0$, choose $M > 1$ such that $1/M < \varepsilon$. Then whenever $A_2 > A_1 > M$,

$$\left|\int_{A_1}^{A_2} \frac{\cos x}{x^2}dx\right| \le \frac{1}{A_1} < \frac{1}{M} < \varepsilon.$$

It follows by the above characterisation theorem that the improper integral converges.

(ii) Suppose on a contrary that the improper integral is convergent. Since $|f| \ge 0$,

$$\int_1^\infty |f(x)|dx \ge \int_\pi^{(N+1)\pi} \frac{|\sin x|}{x}dx = \sum_{k=1}^N \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}dx, \quad \forall N \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, we can substitute $x = t + k\pi$, then

$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}dx = \int_0^\pi \frac{|\sin(t + k\pi)|}{t + k\pi}dt = \int_0^\pi \frac{\sin t}{t + k\pi}dt \ge \frac{1}{(k+1)\pi} \int_0^\pi \sin t dt.$$

Note that $\int_0^\pi \sin t dt = 2$, we have

$$\int_1^\infty |f(x)|dx \ge \sum_{k=1}^N \frac{2}{(k+1)\pi} = \frac{2}{\pi} \sum_{k=1}^N \frac{1}{k+1}, \quad \forall N \in \mathbb{N}.$$

This is a contradiction because the harmonic series diverges to ∞. It follows that the improper integral is divergent.