Fundamental Theorem of Calculus

The most important theorem in calculus is stated below. It relates the concept of differentiation and the concept of integration. Let's first consider the following definitions:

Definition. Let f be Riemann integrable over $[a, b]$.

- A differentiable function F is called an *anti-derivative* of f if

$$F'(x) = f(x), \quad \forall x \in [a, b].$$

- The *indefinite integral* of f with *basepoint* $c \in [a, b]$ is defined by

$$F(x) = \int_c^x f(t)dt.$$

Proposition. *Suppose F_1 and F_2 are anti-derivatives of $f \in \mathcal{R}[a, b]$. Then there exists a constant $c \in \mathbb{R}$ such that $F_1(x) = F_2(x) + c$ for all $x \in [a, b]$.*

Proof. Consider the function $F(x) = F_1(x) - F_2(x)$. Note that

$$F'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0, \quad \forall x \in [a, b].$$

It suffices to show that $F(x) = F(a)$ for all $x \in [a, b]$. In this case, we can take $c = F(a)$.

- If $x = a$, it is obvious.
- If $x \in (a, b]$, note that F is continuous on $[a, x]$ and differentiable on (a, x). By the **Mean Value Theorem**, there exists $\xi \in (a, x)$ such that

$$F(x) - F(a) = F'(\xi)(x - a) = 0.$$

This implies that $F(x) = F(a)$ for all $x \in [a, b]$. The result follows. □

Remark. Any two indefinite integrals with different basepoints also differ by a constant.

Fundamental Theorem of Calculus (c.f. Theorem 2.19 of Lecture Note)**.** *Let $f \in \mathcal{R}[a, b]$.*

(a) If F is an anti-derivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

(b) Let F be an indefinite integral of f. Then F is (Lipschitz) continuous on $[a, b]$. Moreover, if f is continuous at $x \in [a, b]$, then F is differentiable at x and $F'(x) = f(x)$.

Remark. In particular, any indefinite integral of a continuous function is its anti-derivative.

The **Fundamental Theorem of Calculus** helps us evaluate integrals. If an anti-derivative of the integrand is known, we can easily compute the integral.

Example 1. Evaluate $\displaystyle\int_0^\pi \cos x dx$.

Solution. Let $f(x) = \cos x$ and $F(x) = \sin x$. Note that $F'(x) = f(x)$ for all x. Hence F is an anti-derivative of f. It follows from the **Fundamental Theorem of Calculus** that

$$\int_0^\pi \cos x dx = \int_0^\pi f(x)dx = F(\pi) - F(0) = \sin \pi - \sin 0 = 0.$$

Example 2. Observe the following application of the **Fundamental Theorem of Calculus**. Let $f(x) = 1/x^2$ and $F(x) = -1/x$. Note that $F'(x) = f(x)$. Hence

$$\int_{-1}^1 \frac{1}{x^2}dx = \int_{-1}^1 f(x)dx = F(1) - F(-1) = -1 - 1 = -2.$$

This argument is **not correct**. In fact, f is not Riemann integrable over $[-1, 1]$. We cannot apply the **Fundamental Theorem of Calculus** to f.

Example 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $c > 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \int_{x-c}^{x+c} f(t)dt.$$

Show that g is differentiable on $\mathbb{R}$ and find $g'(x)$.

Solution. Since f is continuous, f has an anti-derivative $F : \mathbb{R} \to \mathbb{R}$. Note that

$$g(x) = \int_{x-c}^{x+c} f(t)dt = F(x + c) - F(x - c), \quad \forall x \in \mathbb{R}\,.$$

It follows that $g(x)$ is differentiable on $\mathbb{R}$ with

$$g'(x) = F'(x + c) - F'(x - c) = f(x + c) - f(x - c).$$

Example 4. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Suppose f satisfies

$$\int_0^x f(t)dt = \int_x^1 f(t)dt, \quad \forall x \in [0, 1].$$

Show that $f(x) = 0$ for all $x \in [0, 1]$.

Solution. Since f is continuous, f has an anti-derivative $F : [0, 1] \to \mathbb{R}$. Apply the **Fundamental Theorem of Calculus** to the assumption,

$$F(x) - F(0) = F(1) - F(x) \quad \Longrightarrow \quad F(x) = \frac{F(0) + F(1)}{2}, \quad \forall x \in [0, 1]$$

This implies that F is a constant function. Hence $f(x) = F'(x) = 0$ for all $x \in [0, 1]$.

Riemann Sum

The construction of the Riemann integral from the textbook is different from the Lecture Note. Luckily, they are equivalent.

Theorem (c.f. Theorem 2.23 of Lecture Note)**.** *Let f be a bounded function defined on $[a, b]$. Then $f \in \mathcal{R}[a, b]$ if and only if there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $P = \{x_0, x_1, ..., x_n\}$ is a partition of $[a, b]$ with $\|P\| < \delta$,*

$$\left|\sum_{i=1}^{n} f(\xi_i)\Delta x_i - L\right| < \varepsilon.$$

Here, $\xi_i \in [x_{i-1}, x_i]$ for all $i = 1, 2, ..., n$. In this case, $L = \int_a^b f(x)dx$.

Remark. The sum above is called the **Riemann sum** of f with respect to the tagged partition P with $\xi_i \in [x_{i-1}, x_i]$.

Although this theorem is not useful to determine the integrability of bounded functions, the converse direction is useful to calculate limit of sums in special forms.

Example 5. Evaluate $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2}$.

Solution. Let's show that the limit is $\pi/4$ in a formal way. First notice that for any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \frac{n}{n^2 + k^2} = \sum_{k=1}^{n} \frac{n}{n^2 + k^2} \cdot \frac{1/n^2}{1/n^2} = \sum_{k=1}^{n} \frac{1}{1 + (k/n)^2} \cdot \frac{1}{n}.$$

Consider the function $f(x) = 1/(1 + x^2)$. Then the above sum is the Riemann sum of f with respect to the tagged partition P_n of $[0, 1]$ given by

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\right\} \quad \text{and} \quad \xi_k = \frac{k}{n} \in \left[\frac{k-1}{n}, \frac{k}{n}\right].$$

Hence the limit above is actually given by the integral of f over $[0, 1]$. Notice that f has an anti-derivative $F(x) = \arctan x$. By the **Fundamental Theorem of Calculus**,

$$\int_0^1 f(x)dx = F(1) - F(0) = \arctan 1 - \arctan 0 = \frac{\pi}{4}.$$

To show that the limit is $\pi/4$, let $\varepsilon > 0$. By the theorem above, there exists $\delta > 0$ such that whenever $P = \{x_0, x_1, ..., x_m\}$ is a partition of $[0, 1]$ with $\|P\| < \delta$,

$$\left|\sum_{i=1}^{m} f(\xi_i)\Delta x_i - \frac{\pi}{4}\right| < \varepsilon,$$

where $\xi_i \in [x_{i-1}, x_i]$ for all $i = 1, 2, ..., m$. Now choose $N \in \mathbb{N}$ so that $1/N < \delta$. Whenever $n \geq N$, consider the partition P_n and ξ_i defined as above. Then $\|P_n\| = 1/n < \delta$. Hence

$$\left|\sum_{k=1}^{n} \frac{n}{n^2 + k^2} - \frac{\pi}{4}\right| = \left|\sum_{k=1}^{n} f(\xi_k)\Delta x_k - \frac{\pi}{4}\right| < \varepsilon.$$

We usually don't need to present the evaluation of Riemann sums as rigorous as above.

Example 6. Evaluate

(a) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\pi\right)$ (b) $\displaystyle\lim_{n\to\infty}\left(\frac{n+1}{n^2}+\frac{n+2}{n^2}+\cdots+\frac{2n}{n^2}\right)$

Solution. We just need to transform the sum into the Riemann sum of a suitable function and then evaluate the desired integral.

(a) Note that for each $n\in\mathbb{N}$,

$$\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\pi\right)=\frac{1}{n}\sum_{k=1}^{n}\sin\frac{k\pi}{n}=\sum_{k=1}^{n}\sin\left(\frac{k}{n}\pi\right)\cdot\frac{1}{n}.$$

Hence the sum is the Riemann sum of the function $\sin(\pi x)$. Therefore

$$\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\pi\right)=\int_0^1\sin(\pi x)dx=-\frac{1}{\pi}\cos(\pi x)\bigg|_{x=0}^{x=1}=\frac{2}{\pi}.$$

(b) Note that for each $n\in\mathbb{N}$,

$$\frac{n+1}{n^2}+\frac{n+2}{n^2}+\cdots+\frac{2n}{n^2}=\sum_{k=1}^{n}\frac{n+k}{n^2}=\sum_{k=1}^{n}\frac{n+k}{n}\cdot\frac{1}{n}=\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)\cdot\frac{1}{n}.$$

Hence the sum is the Riemann sum of the function $1+x$. Therefore

$$\lim_{n\to\infty}\left(\frac{n+1}{n^2}+\frac{n+2}{n^2}+\cdots+\frac{2n}{n^2}\right)=\int_0^1(1+x)dx=x+\frac{1}{2}x^2\bigg|_{x=0}^{x=1}=\frac{3}{2}.$$