The Riemann Integral

Definition (Partition)**.** Let $I = [a, b]$ be a **closed and bounded** interval.

- A *partition* P of $[a, b]$ is a **finite set** of points $x_0, x_1, ..., x_n \in [a, b]$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

- For each $i = 1, 2, ..., n$, denote $\Delta x_i = x_i - x_{i-1}$.
- The *norm* of P is denoted by $\|P\| = \max \Delta x_i$.

Definition (Upper sum and Lower sum)**.** Let f be a **bounded** function defined on a closed and bounded interval $[a, b]$, and let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$.

- For each $i = 1, 2, ..., n$, denote

$$m_i(f, P) = \inf \Big\{ f(x) : x \in [x_{i-1}, x_i] \Big\} \quad \text{and} \quad M_i(f, P) = \sup \Big\{ f(x) : x \in [x_{i-1}, x_i] \Big\}.$$

- The *lower sum* and *upper sum* of f with respect to P are denoted respectively by

$$L(f, P) = \sum_{i=1}^{n} m_i(f, P)\Delta x_i \quad \text{and} \quad U(f, P) = \sum_{i=1}^{n} M_i(f, P)\Delta x_i.$$

Remark. Since f is a bounded function, we can define

$$m = \inf \Big\{ f(x) : x \in [a, b] \Big\} \quad \text{and} \quad M = \sup \Big\{ f(x) : x \in [a, b] \Big\}.$$

Then for each $i = 1, 2, ..., n$, the set $\Big\{ f(x) : x \in [x_{i-1}, x_i] \Big\}$ is bounded between m and M. Therefore m_i and M_i are well-defined. Moreover, we always have the following estimation:

$$m(b - a) \le L(f, P) \le U(f, P) \le M(b - a)$$

Furthermore, the oscillations $\omega_i(f, P) = M_i(f, P) - m_i(f, P)$ can be given by

$$\omega_i(f, P) = \sup \Big\{ |f(x) - f(y)| : x, y \in [x_{i-1}, x_i] \Big\}.$$

The following observation highlight the important features of the above definitions.

Lemma (c.f. Lemma 2.2 of Lecture Note)**.** *Let f be a bounded function defined on a closed and bounded interval $[a, b]$, and let P, Q be partitions of $[a, b]$.*

(a) If $P \subseteq Q$, then $L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P)$.

(b) $L(f, P) \le U(f, Q)$.

Remark. Q is called a *refinement* of P if $P \subseteq Q$.

Definition. Let f be a bounded function defined on a closed and bounded interval $[a, b]$.

- The *lower integral* and *upper integral* of f are denoted respectively by

$$\underline{\int_a^b} f = \sup\Big\{L(f, P) : P \text{ is a partition of } [a, b]\Big\}$$

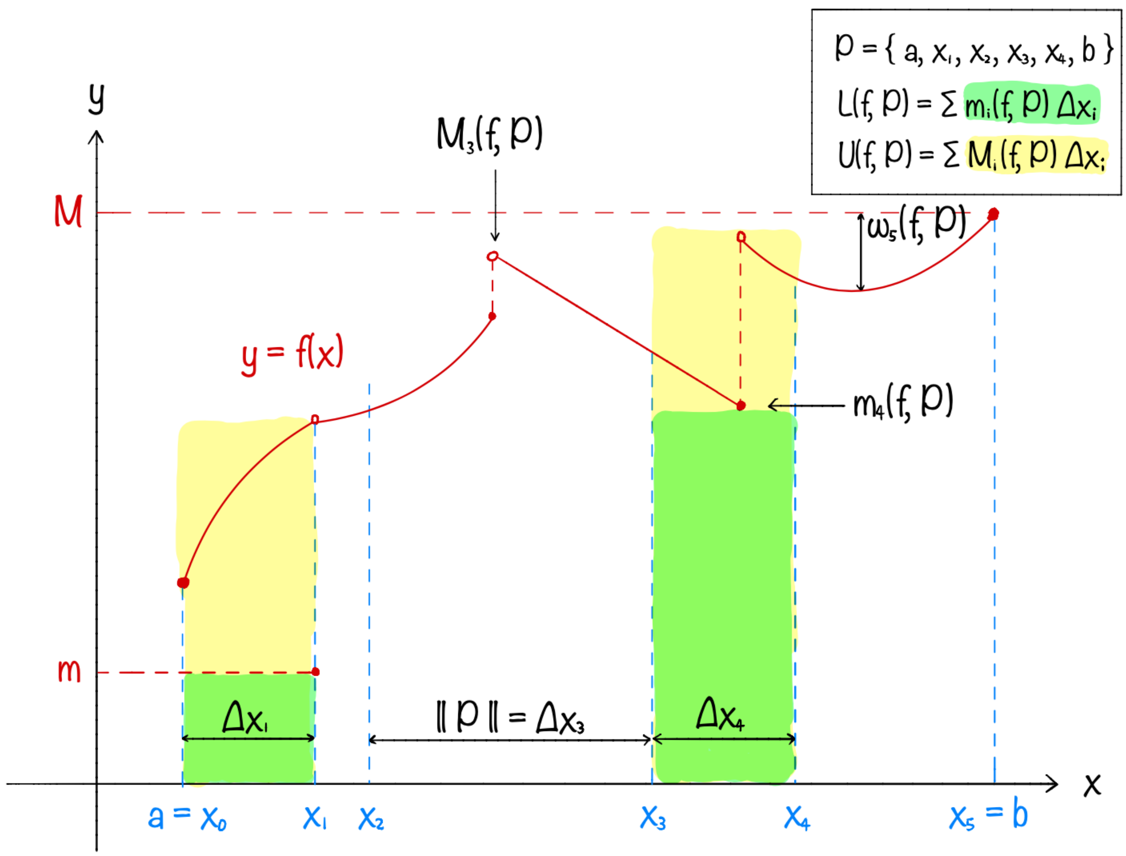
$$\overline{\int_a^b} f = \inf\Big\{U(f, P) : P \text{ is a partition of } [a, b]\Big\}$$

- f is said to be *Riemann integrable* over $[a, b]$ if the lower integral and upper integral are equal. In this case, the *integral* of f, denoted by

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)dx,$$

is defined as their common value and we write $f \in \mathcal{R}[a, b]$.

Example 1. The above quantities are visualized geometrically in the figure below:



Theorem (c.f. Theorem 2.10 of Lecture Note)**.** *Let f be a bounded function defined on a closed and bounded interval $[a, b]$. f is Riemann integrable over $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that*

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} \omega_i(f, P)\Delta x_i < \varepsilon.$$

Example 2. Show that the identity function $f(x) = x$ is Riemann integrable over $[0, 1]$.

Solution. Let's calculate the lower and upper integrals of f. Fix any $n \in \mathbb{N}$ and consider the partition P_n of $[0, 1]$ given by

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\right\}.$$

Notice that for each $i = 1, 2, ..., n$,

$$\Delta x_i = \frac{i}{n} - \frac{i-1}{n} = \frac{1}{n}, \quad m_i(f, P_n) = \frac{i-1}{n}, \quad \text{and} \quad M_i(f, P_n) = \frac{i}{n}.$$

Hence the lower and upper sums of f with respect to P_n are

$$L(f, P_n) = \sum_{i=1}^{n} m_i(f, P_n)\Delta x_i = \sum_{i=1}^{n} \frac{i-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \frac{n(n-1)}{2} = \frac{1}{2}\left(1 - \frac{1}{n}\right)$$
$$U(f, P_n) = \sum_{i=1}^{n} M_i(f, P_n)\Delta x_i = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{1}{2}\left(1 + \frac{1}{n}\right)$$

It follows that the lower and upper integrals of f satisfy:

$$\frac{1}{2}\left(1 + \frac{1}{n}\right) = L(f, P_n) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f, P_n) = \frac{1}{2}\left(1 + \frac{1}{n}\right).$$

Since $n \in \mathbb{N}$ is arbitrary, taking limit on both sides gives

$$\frac{1}{2} \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le \frac{1}{2}.$$

It follows that the upper and lower integrals of f are equal to each other. i.e., $f \in \mathcal{R}[0, 1]$.

Remark. The integral of f over $[0, 1]$ is $1/2$.

Example 3. Show that the Dirichlet function is not Riemann integrable over $[0, 1]$.

Solution. Recall that the Dirichlet function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Consider any partition P of $[0, 1]$. Notice that for each $i = 1, 2, ..., n$,

$$m_i(f, P) = 0, \quad \text{and} \quad M_i(f, P) = 1.$$

Hence the lower and upper sums of f with respect to P are

$$L(f, P) = \sum_{i=1}^{n} m_i(f, P_n)\Delta x_i = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0$$
$$U(f, P) = \sum_{i=1}^{n} M_i(f, P_n)\Delta x_i = \sum_{i=1}^{n} 1 \cdot \Delta x_i = 1$$

Since P is an arbitrary partition of $[0,1]$, taking supremum and infimum yields that the lower and upper integrals of f are

$$\underline{\int_a^b} f = 0 \quad \text{and} \quad \overline{\int_a^b} f = 1.$$

It follows that the upper and lower integrals of f are not equal to each other. i.e., $f \notin \mathcal{R}[0,1]$.

Example 4. Let f be a continuous and non-negative function defiend on $[a,b]$. Show that $f(x) = 0$ for all $x \in [a,b]$ if

$$\int_a^b f = 0.$$

Solution. Suppose on a contrary that $f(c) > 0$ for some $c \in [a,b]$. By the continuity of f, there exists $\delta > 0$ such that whenever $x \in [a,b]$ and $|x - c| < \delta$,

$$|f(x) - f(c)| < \frac{1}{2}f(c) \quad \implies \quad 0 < \frac{1}{2}f(c) < f(x) < \frac{3}{2}f(c).$$

Suppose $c \in (a,b)$. (The special cases $c = a$ and $c = b$ are left as an exercise.) Let u, v be points in $[a,b]$ satisfying

$$c - \delta < u < v < c + \delta.$$

Consider the partition P of $[a,b]$ given by

$$P = \{a, u, v, b\}.$$

Then the lower sum of f with respect to P can be estimated by:

$$\begin{aligned} L(f,P) &= m_1(f,P)\Delta x_1 + m_2(f,P)\Delta x_2 + m_3(f,P)\Delta x_3 \\ &\geq 0 \cdot (u-a) + \frac{1}{2}f(c) \cdot (v-u) + 0 \cdot (b-v) \\ &> 0 \end{aligned}$$

It follows that

$$0 = \int_a^b f = \underline{\int_a^b} f \geq L(f,P) > 0.$$

This is a contradiction, so f is constantly zero.