General Information

- Textbook: *Introduction to Real Analysis* by Robert G. Bartle, Donald R. Sherbert. (Try to google the title of the textbook for MORE information!)
- This course is a continuation of MATH2050 Mathematical Analysis I, which the materials of this course depend heavily on. Make sure that you are familiar with them.
- Please visit the course web-page at `https://www.math.cuhk.edu.hk/course/2021/math2060b` **frequently** to get the most updated information. It shall contain the information for homework and tests, as well as lecture notes and tutorial notes.
- I am the tutor of this section. You may call me **Ernest**. You are welcomed to send me an email if you need help. My email address is **ylfan@math.cuhk.edu.hk**.
- The grader of this section is **Marco**. He is responsible for grading the assignments. His email address is **kllam@math.cuhk.edu.hk**.
- Please submit each of your homework and tests in **one PDF file** to **Blackboard**.
- The two tutorials in the same week contain the same content, you may attend either class on your choice.

Differentiability

Definition (c.f. Definition 6.1.1)**.** Let $I \subseteq \mathbb{R}$ be an interval containing c and $f : I \to \mathbb{R}$ be a function. A real number L is said to be the *derivative* of f at c if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{f(x)-f(c)}{x-c} - L\right| < \varepsilon, \quad \text{whenever } x \in I \text{ and } 0 < |x-c| < \delta.$$

In this case, we say that f is *differentiable* at c and denote $f'(c) = L$.

Remark. By comparing to the definitions of differentiability and the limit of functions, f is differentiable at c with derivative L if and only if the following limits exist and equal L:

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

We can actually define the derivative of a function at limit points in an arbitrary subsets of $\mathbb{R}$, but the concept is more natural if we restrict the attention on intervals.

The following theorem can be derived easily from definition.

Theorem (c.f. Theorem 6.1.2)**.** *Let $I \subseteq \mathbb{R}$ be an interval containing c. If $f : I \to \mathbb{R}$ is differentiable at c, then f is continuous at c.*

Example 1. Show that the following functions are differentiable at $x = 1$ and find their respecitve derivatives:

(a) $f(x) = x^2$ (b) $f(x) = \sqrt{x}$ (c) $f(x) = \sin x$

Solution. We will use different approaches to show the results.

(a) Let's use ε-δ notation to show that the derivative of f at $x = 1$ is 2. Note that

$$\left|\frac{x^2 - 1^2}{x - 1} - 2\right| = \left|\frac{x^2 - 1 - 2(x-1)}{x - 1}\right| = \left|\frac{(x-1)^2}{x-1}\right| = |x - 1|, \quad \forall x \neq 1.$$

Let $\varepsilon > 0$ and take $\delta = \varepsilon$. Then whenever $0 < |x - 1| < \delta$,

$$\left|\frac{x^2 - 1^2}{x - 1} - 2\right| = |x - 1| < \delta = \varepsilon.$$

Hence $f'(1) = 2$, so f is differentiable at $x = 1$ with derivative 2.

(b) Let's use limit notation to show that the derivative of f at $x = 1$ is 0.5. Note that

$$\frac{\sqrt{x} - \sqrt{1}}{x - 1} = \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \frac{1}{\sqrt{x} + 1}, \quad \forall x \neq 1.$$

Hence we have

$$\lim_{x \to 1} \frac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{\sqrt{1} + 1} = \frac{1}{2}.$$

Hence $f'(1) = 0.5$, so f is differentiable at $x = 1$ with derivative 0.5.

(c) Again, we use limit notation to find the derivative. Note that

$$\frac{\sin(1 + h) - \sin 1}{h} = \frac{2}{h} \cos\left(\frac{2 + h}{2}\right) \sin\frac{h}{2} = \cos\left(\frac{2 + h}{2}\right) \cdot \frac{\sin(h/2)}{h/2}, \quad \forall h \neq 0.$$

Using continuity and the fact that $(\sin x)/x \to 1$ as $x \to 0$, we have

$$\lim_{h \to 0} \frac{\sin(1 + h) - \sin 1}{h} = \cos\left(\frac{2 + 0}{2}\right) \cdot \lim_{h \to 0} \frac{\sin(h/2)}{h/2} = \cos 1 \cdot 1 = \cos 1.$$

Hence $f'(1) = 1$, so f is differentiable at $x = 1$ with derivative 1.

Example 2. Show that the following functions are not differentiable at $x = 0$:

(a) $f(x) = |\sin x|$, (b) $f(x) = \sqrt[3]{x}$, (c) $f(x) = \operatorname{sgn}(x)$.

Solution. We can observe the graphs of the functions and see that they are not differentiable.

(a) We show that the following limit does not exists:

$$\lim_{x\to 0} \frac{|\sin x| - |\sin 0|}{x-0} = \lim_{x\to 0} \frac{|\sin x|}{x}$$

This can be done by looking at its left-hand and right-hand limits:

$$\lim_{x\to 0^-} \frac{|\sin x|}{x} = -\lim_{x\to 0^-} \frac{\sin x}{x} = -1 \quad \text{and} \quad \lim_{x\to 0^+} \frac{|\sin x|}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

Since the one-sided limits are not equal, the limit does not exist. Hence f is not differentiable at $x = 0$.

(b) We show that the following limit does not exists:

$$\lim_{x\to 0} \frac{\sqrt[3]{x} - 0}{x-0} = \lim_{x\to 0} \frac{x^{1/3}}{x} = \lim_{x\to 0} x^{-2/3}$$

Recall that if the limit exists, then $x^{-2/3}$ must be bounded in some neighbourhood of 0 (c.f. **Theorem 4.2.2**). i.e., there exists $\delta > 0$ and $M > 0$ such that

$$|x^{-2/3}| \leq M, \quad \text{whenever } 0 < |x| < \delta.$$

However, we can always find a small enough number x_0 such that $0 < x_0 < \delta$ and $x_0^{-2/3} > M$. Therefore the limit does not exists, so f is not differentiable at $x = 0$.

(c) Notice that f is not continuous at $x = 0$. Hence it is not differentiable at $x = 0$ by **Theorem 6.1.2**.

Theorem (c.f. Theorem 6.1.3)**.** *Let $I \subseteq \mathbb{R}$ be an interval containing c, $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at c. Then:*

(a) For any $\alpha \in \mathbb{R}$, the function αf is differentiable at c, and

$$(\alpha f)'(c) = \alpha f'(c).$$

(b) The function $f + g$ is differentiable at c, and

$$(f+g)'(c) = f'(c) + g'(c).$$

*(c) (**Product Rule**) The function fg is differentiable at c, and*

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

*(d) (**Quotient Rule**) If $g(c) \neq 0$, then the function f/g is differentiable at c, and*

$$(f/g)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}.$$

Chain Rule

The following theorem characterize the differentiability of a function at a certain point. It is essential to prove the chain rule.

Carathéodory's Theorem (c.f. 6.1.5). *Let f be defined on an interval I containing the point c. Then f is differentiable at c if and only if there exists a function φ on I that is continuous at c and*

$$f(x) - f(c) = \varphi(x)(x - c), \quad \forall x \in I.$$

In this case, we have $\varphi(c) = f'(c)$.

Chain Rule (c.f. 6.1.6). *Let I, J be intervals in $\mathbb{R}$, let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be functions such that $f(J) \subseteq I$, and let $c \in J$. If f is differentiable at c and if g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and*

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Remark. The basic results of differentiability are clear. Please focus on their proofs.

Example 3 (c.f. Section 6.1, Ex.12). If $r > 0$ is a rational number, let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^r \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Determine those values of r for which $f'(0)$ exists.

Solution. We show that $f'(0)$ exists if and only if $r > 1$. First note that the difference quotient at 0 is given by

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^r \sin(1/x) - 0}{x - 0} = x^{r-1} \sin(1/x), \quad \forall x > 0.$$

Hence we have

$$-x^{r-1} \leq \left| \frac{f(x) - f(0)}{x - 0} \right| \leq x^{r-1}, \quad \forall x > 0.$$

- If $r > 1$, notice that $x^{r-1} \to 0$ as $x \to 0^+$. Hence by the **Squeeze Theorem**,

  $$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

- If $r = 1$, consider the sequences (x_n) and (u_n) in $[0, \infty)$ defined by

  $$x_n = \frac{1}{2n\pi + \pi/2} \quad \text{and} \quad u_n = \frac{1}{2n\pi}.$$

  Then we have $x_n \to 0$ and $u_n \to 0$, and

  $$\lim_{n \to \infty} \frac{f(x_n) - f(0)}{x_n - 0} = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{f(u_n) - f(0)}{u_n - 0} = 0.$$

  So the limit of the difference quotient does not exists. i.e., $f'(0)$ does not exists.

- If $0 < r < 1$, consider the sequence (x_n) defined as above. We have $x_n \to 0$ and

  $$\frac{f(x_n) - f(0)}{x_n - 0} = \left(2\pi n + \frac{\pi}{2}\right)^{1-r}.$$

  Note that this sequence is unbounded, so the limit of the difference quotient does not exists. i.e., $f'(0)$ does not exists.