MATH 2060B - HW 5 - Solutions[1]

1 (P.224 Q14)**.** Show that there does NOT exist a function $f \in \mathcal{C}^1[0,2]$ (cotinuously differentiable on $[0,2]$) such that

i. $f(0) = -1$

ii. $f(2) = 4$

iii. $f'(x) \leq 2$ for all $x \in [0,2]$

(The Fundamental Theorem may be useful).

Solution.
Method 1: Using Fundamental Theorem of Calculus (FTC):
Suppose not. Let $f \in \mathcal{C}^1[0,2]$ satisfying the conditions. Then f' is continuous on $[0,2]$. Note that f is an anti-derivative of f' by definition. It follows from part (i) of FTC as stated in the Lecture Note that $f(2) - f(0) = \int_0^2 f'(x)dx$. It follows that

$$|f(2) - f(0)| = \left| \int_0^2 f'(x)dx \right| \leq \int_0^2 |f'(x)|dx \leq \int_0^2 2dx = 4$$

However, we have $f(2) - f(0) = 4 - (-1) = 5$, which contradicts the above inequality.

Method 2: Using Mean Value Theorem:
Suppose not. Note that f is differentiable on $(0,2)$ and continuous on $[0,2]$ by assumption. Hence by the Mean Value Theorem, there exists $\xi \in (0,2)$ such that $f(2) - f(0) = f'(\xi)(2-0)$, which implies $4 - (-1) = 2f'(\xi)$. Hence, $f'(\xi) = 2.5 > 2$, contradicting the third assumption.

Comment. The equality $f(2) - f(0) = \int_0^2 f'(x)dx$ follows from the continuity (or integrability) of f' by FTC part (i).

[1] Please feel free to email your TA at kllam@math.cuhk.edu.hk for any questions concerning homework.

2 (P.224 Q17)**.** Let $J := [\alpha, \beta]$ and let $\phi : J \to \mathbb{R}$ be cotinuously differentiable on J. Let $f : I \to R$ be continuous on an interval I such that $\phi(J) \subset I$.
Define $F(u) := \int_{\phi(\alpha)}^{u} f(x)dx$ for all $u \in I$ and $H(t) := F(\phi(t))$ for all $t \in J$.

i. Show that $H'(t) = f(\phi(t))\phi'(t)$ for all $t \in J$

ii. Show that

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = F(\phi(\beta)) = H(\beta) = \int_{\alpha}^{\beta} f(\phi(t))\phi'(t)dt$$

Remark. This is a proof to the Substitution Theorem (Theorem 7.3.8)

Solution.

i. Consider I to be an open interval and write $I = (a, b)$ where $a < b$. Since f is continuous on I, it is Riemann-integrable on I and so on $[a, b]$ (why?). We can define $F_0(x) := \int_a^x f(t)dt$ for all $x \in [a, b]$. Note that even though f is only known to be continuous on (a, b) without knowledge of continuity on endpoints, it is still easy to see that $F_0'(x) = f'(x)$ for all $x \in (a, b)$:
Let $x \in (a, b)$ and let $\epsilon > 0$. Then by continuity of f, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $y \in B_\delta(x)$. It then follows that for all $y \in B_\delta(x) \backslash \{0\} \cap (a, b)$, we have

$$\frac{F_0(y) - F_0(x)}{y - x} = \frac{1}{y - x}\int_x^y f(t)dt = \frac{1}{y - x}\int_x^y f(t) - f(x)dt + f(x)$$

Hence, for all $y \in B_\delta(x) \backslash \{0\} \cap (a, b)$, we have

$$\begin{aligned}\left|\frac{F_0(y) - F_0(x)}{y - x} - f(x)\right| &= \frac{1}{|y - x|}\left|\int_x^y f(t) - f(x)dt\right| \\ &\leq \frac{1}{|y - x|}\int_x^y |f(t) - f(x)|dt \leq \frac{1}{|y - x|} \cdot |y - x| \cdot \epsilon\end{aligned}$$

It follows by definition that $F_0'(x) := \lim_{y \to x} \frac{F(y) - F(x)}{y - x} = f'(x)$

Note that $F(u) = \int_{\phi(\alpha)}^u f(x)dx = \int_a^u f(x)dx - \int_a^{\phi(\alpha)} f(x)dx = F_0(u) + C$ for all $u \in I$ where C is some constant. Hence it follows that F is differentiable on I with $F'(u) = F_0'(u) = f(u)$ for all $u \in I$.
In addition by assumption, we have that ϕ is differentiable on J and $\phi(J) \subset I$.
By the chain rule, it follows that $H(t) := F \circ \phi(t)$ is differentiable for all $t \in J$. Furthermore, we have

$$H'(t) = F'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t)$$

for all $t \in J$.
The proof for non-open interval is similar where differentiability at end-points is defined via one-sided limit.

ii. Since ϕ is continuously differentiable, ϕ, ϕ' are continuous. Furthermore f is continuous by assumption. It follows that H' is continuous on J by the formula on (i). It follows from the Fundamental Theorem of Calculus that

$$H(\beta) - H(\alpha) = \int_\alpha^\beta H'(t)dt = \int_\alpha^\beta f(\phi(t))\phi'(t)dt$$

while by definitions, we have

$$H(\beta) - H(\alpha) = F(\phi(\beta)) - F(\phi(\alpha)) = \int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx - \int_{\phi(\alpha)}^{\phi(\alpha)} f(x)dx = \int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx$$

The result follows by noting that $H(\alpha) = F(\phi(\alpha)) = 0$.

Comment.

i. For part (i), the conditions in (i) are not quite the same as those stated in the Lecture Notes on the Fundamental Theorem of Calculus. Here f is not necessarily continuous on a *closed bounded* interval and we also need H' to exist on a *closed bounded* interval instead of an open interval. The first paragraph of the solution justifies the validity of FTC under the slightly modified assumptions and conclusions.

ii. The differentiablity of F follows from the continuity of f by FTC part (ii) as in the Lecture Note.

iii. The equanlity $H(\beta) - H(\alpha) = \int_\alpha^\beta H'(t)dt$ follows from the continuity (in fact also integrability) of H' by FTC part (i) as in the Lecture Note.

3 (P.224 Q22)**.** Let $h : [0,1] \to \mathbb{R}$ be the Thomae's function, that is

$$h(x) = \begin{cases} \frac{1}{p} & x = \frac{q}{p}, \gcd(q,p) = 1, q, p \in \mathbb{N} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

for all $x \in [0,1]$. Let sgn be the sign function. Show that the composition $\operatorname{sgn} \circ h$ is not Riemann integrable on $[0,1]$.

Solution. Note that $g := \operatorname{sgn} \circ h(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ for all $x \in [0,1]$.
We can show g is not Riemann integrable on $[0,1]$ by showing that $\overline{\int}_0^1 g(x)dx = 1$ and $\underline{\int}_0^1 g(x)dx = 0$. By denseness of $\mathbb{Q}$ and $\mathbb{Q}^c$ resepctively, it is easy to see that $U(g,P) = 1$ and $L(g,P) = 0$ for all partition P.
The result follows clearly taking infimum and supremum on upper and lower sums respectively.

Comment. Note that both the Thomae's function and the sign function are Riemman integrable on $[0,1]$. The former can be proved by noting that the Thomae's function is continuous except for a countable set (the function is continuous on irrational points). The question thus shows that composition of Riemann integrable functions may not be Riemann integrable.