MATH 2060B - HW 4 - Solutions[1]

1 (P.215 Q2)**.** Let $h : [0,1] \to \mathbb{R}$ be defined by $h(x) := \begin{cases} x+1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \notin \mathbb{Q} \cap [0,1] \end{cases}$. Show that h is not Riemann integrable.

Solution. Recall that a bounded function f is Riemann integrable if and only if $\overline{\int_0^1} f = \underline{\int_0^1} f$. Clearly h is bounded. It remains to show that $\overline{\int_0^1} h \neq \underline{\int_0^1} h$.

Claim 1: $\overline{\int_0^1} h \geq 1$.
Let $P := \{x_0 := 0, \cdots, x_n := 1\}$ be a partition (where $x_{i-1} < x_i$ for all $i = 1, \cdots, n$). Note that for all $i = 1, \cdots, n$, by denseness of $\mathbb{Q}$, we have $\mathbb{Q} \cap (x_{i-1}, x_i) \neq \phi$. Hence by considering some $q_i \in \mathbb{Q} \cap (x_{i-1}, x_i) \subset [x_{i-1}, x_i]$, it follows that $M_i(h,P) := \sup_{t \in [x_{i-1}, x_i]} h(t) \geq h(q_i) = q_i + 1 \geq 1$. Therefore, we have $U(h,P) := \sum_{i=1}^n M_i(h,P)(x_i - x_{i-1}) \geq \sum_{i=1}^n 1 \cdot (x_i - x_{i-1}) = 1$ for all partition P. By taking infermum, it is clear that $\overline{\int_0^1} h := \inf_P U(h,P) \geq 1$.
Claim 2: $\underline{\int_0^1} h \leq 0$.
The proof is almost the same to that of Case 1. Let $P := \{x_0 := 0, \cdots, x_n := 1\}$ be a partition (where $x_{i-1} < x_i$ for all $i = 1, \cdots, n$). Note that for all $i = 1, \cdots, n$, by denseness of $\mathbb{Q}^c$, we have $\mathbb{Q}^c \cap (x_{i-1}, x_i) \neq$. Hence by considering some $\alpha_i \in \mathbb{Q}^c \cap (x_{i-1}, x_i) \subset [x_{i-1}, x_i]$, it follows that $m_i(h,P) := \inf_{t \in [x_{i-1}, x_i]} h(t) \leq h(\alpha_i) = 0$. Therefore, we have $L(h,P) = \sum_{i=1}^n m_i(h,P)(x_i - x_{i-1}) \leq \sum_{i=1}^n 0 \cdot (x_i - x_{i-1}) = 0$ for all partition P. By taking supremum, it is clear that we have $\underline{\int_0^1} h := \sup_P L(h,P) \leq 0$.
Combining the two claims, we have that $\overline{\int_0^1} h \neq \underline{\int_0^1} h$. The result follows.

Comment. It is more desirable for you to use the characterizations of integrability *mentioned in the Lecture Notes* instead of the textbook. The former utilizes the approach of Darboux, which uses lower/upper sum, while the latter utilizes that of Riemann, which uses tagged partitions.

[1]Please feel free to email your TA at kllam@math.cuhk.edu.hk for any questions concerning homework.

2 (P.215 Q8)**.** Let f be continuous on $[a,b]$ $(a,b \in \mathbb{R})$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Solution. Suppose not. There existsed $x_0 \in [a,b]$ such that $f(x_0) > 0$. By continuity of f at x_0, there exists $r > 0$ such that $f(x) > f(x_0)/2$ for all $x \in I_{x_0} := (x_0 - r, x_0 + r) \cap [a,b]$ (why?). Note that I_{x_0} is an interval with length $\geq r$. Without loss of generality, write $I_{x_0} := (c,d)$ to be an open interval. Then by spliting the domain of f (Proposition 2.15) together with the order-preserving property of the integral operator (and the integrability of constant functions), we have

$$\begin{aligned}\int_a^b f &= \int_a^c f + \int_c^d f + \int_d^b f \\ &\geq \int_a^c 0 + \int_c^d \frac{f(x_0)}{2} + \int_d^b 0 \\ &= 0 + \frac{f(x_0)}{2}(d-c) + 0 \\ &\geq \frac{f(x_0)}{2} r > 0\end{aligned}$$

where we use the convention that $\int_x^y f = 0$ when $x = y \in \mathbb{R}$. It contradicts to the assumption that $\int_a^b f = 0$. It follows that $f(x) = 0$ for all $x \in [a,b]$

Comment. The continuity of f is crucial. It is easy to construct a discontinuous example violating the conclusion, for example consider the characteristic function of $\{0\}$ on $[0,1]$, $\chi_{\{0\}}(x) := \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

3 (P.215 Q12)**.** Define $g : [0,1] \to \mathbb{R}$ by $g(x) := \begin{cases} \sin(1/x) & x \in (0,1] \\ 0 & x = 0 \end{cases}$ Show that $g \in \mathcal{R}[0,1]$

Solution. This solution makes heavy use of Theorem 2.10.
Let $1 > \epsilon > 0$ (why does it suffice to pick an ϵ bounded by 1?). We want to find a partition P of $[0,1]$ such that $\sum_{i=1}^{n} \omega_i(g,P)(x_{i-1} - x_i) < \epsilon$ where $\omega_i(g,P) := \sup_{s,t \in [x_{i-1},x_i]} |g(s) - g(t)|$.
First note that $g(x) = \sin(1/x)$ on $[\epsilon, 1]$. Hence g is continuous on $[\epsilon, 1]$ and so is Riemann integrable on $[\epsilon, 1]$. Then by the characterization of Riemann integrability, there exists a partition defined by $P_\epsilon := \{y_0 := \epsilon, \cdots, y_n := 1\}$ of $[\epsilon, 1]$ such that $\sum_{i=1}^{n} \omega_i(g, P_\epsilon)(y_{i-1} - y_i) < \epsilon$. Now consider the partition P on $[0,1]$ given by $x_0 := 0, x_1 := y_0 = \epsilon, \cdots, x_{i+1} = y_i$ for all $i = 0, \cdots, n$. It follows that we have

$$\begin{aligned}
\sum_{i=1}^{n+1} \omega_i(g,P)(x_{i-1} - x_i) &= \omega_0(g,P)(x_1 - x_0) + \sum_{i=2}^{n+1} \omega_i(g, P_\epsilon)(x_{i-1} - x_i) \\
&= \sup_{s,t \in [0,\epsilon]} |g(s) - g(t)|\epsilon + \sum_{i=1}^{n} \omega_i(g, P_\epsilon)(y_{i-1} - y_i) \\
&\leq 2M\epsilon + \epsilon = (2M+1)\epsilon
\end{aligned}$$

where M is some upper bound of $|g|$ on $[0,1]$ (for example we can take $M = 2$). The result follows from the characterization of Riemann integrability.

Comment. The above proof shows directly that $g \in \mathcal{R}[0,1]$. It does not show that $g \in \mathcal{R}[0,\epsilon]$ (why?) and so it is wrong to say $g \in \mathcal{R}[0,1]$ *because* we have $g \in \mathcal{R}[0,\epsilon]$ and $g \in \mathcal{R}[\epsilon, 1]$.