MATH 2060B - Home Test 2

Suggested Solutions*(It does not reflect the marking scheme)*

1. (15 points)

Let f be a C^1-function defined on $(0,\infty)$. For $x \in \mathbb{R}$, let $[x]$ denote the greatest integer not greater than x.

(i) Show that if a and b are positive integers with $a < b$, then

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)dx + \int_a^b f'(x)(x - [x] - \frac{1}{2})dx + \frac{f(a)+f(b)}{2}.$$

(ii) Show that if $p \neq 1$, then

$$\sum_{k=1}^{n} \frac{1}{k^p} = \frac{1}{n^{p-1}} + p\int_1^n \frac{[x]}{x^{p+1}}dx.$$

Solution.

(i) Consider the integrals on the right hand side of the equation:

$$\int_a^b f(x)dx + \int_a^b f'(x)(x - [x] - \frac{1}{2})dx = \sum_{k=a}^{b-1} \int_k^{k+1} \left[f(x) + f'(x)(x - [x] - \frac{1}{2}) \right] dx$$

We need to calculate the sum of

$$A_k = \int_k^{k+1} \left[f(x) + f'(x)(x - [x] - \frac{1}{2}) \right] dx.$$

Notice that $[x] = k$ for all $x \in [k, k+1]$ only except for $x = k+1$. Hence

$$A_k = \int_k^{k+1} (f(x) + xf'(x))dx - (k + \frac{1}{2}) \int_k^{k+1} f'(x)dx.$$

By the **product rule**, $[xf(x)]' = f(x) + xf'(x)$. Then apply the **Fundamental Theorem of Calculus**, we have

$$A_k = \left[xf(x) - (k + \frac{1}{2})f(x) \right]_{x=k}^{x=k+1} = \frac{1}{2}f(k+1) + \frac{1}{2}f(k).$$

It follows that

$$\sum_{k=a}^{b-1} A_k = \sum_{k=a}^{b-1} \left(\frac{1}{2}f(k+1) + \frac{1}{2}f(k) \right) = \frac{1}{2}f(a) + f(a+1) + \cdots + f(b-1) + \frac{1}{2}f(b).$$

Hence we have

$$\begin{aligned} \sum_{k=a}^{b} f(k) &= \left(\frac{1}{2}f(a) + f(a+1) + \cdots + f(b-1) + \frac{1}{2}f(b) \right) + \frac{f(a)+f(b)}{2} \\ &= \int_a^b f(x)dx + \int_a^b f'(x)(x - [x] - \frac{1}{2})dx + \frac{f(a)+f(b)}{2} \end{aligned}$$

(ii) If $n = 1$, then both sides of the equation equal to 1. If $p = 0$, then both sides of the equation equal to n. It remains to show the cases $n > 1$ and $p \neq 0, 1$. Consider the function defined by $f(x) = 1/x^p$, which is $\mathcal{C}^1$ on $(0, \infty)$. Note that

$$f'(x) = -p \cdot \frac{1}{x^{p+1}}, \quad \forall x \in (0, \infty).$$

Using (i), we have

$$\sum_{k=1}^{n} \frac{1}{k^p} = \int_1^n \frac{1}{x^p} dx - p \int_1^n \frac{1}{x^{p+1}} (x - [x] - \frac{1}{2}) dx + \frac{1 + 1/n^p}{2}.$$

We proceed to calculate

$$\int_1^n \frac{1}{x^p} dx - p \int_1^n \frac{1}{x^{p+1}} (x - \frac{1}{2}) dx = (1 - p) \int_1^n \frac{1}{x^p} dx + \frac{p}{2} \int_1^n \frac{1}{x^{p+1}} dx.$$

Notice that we have

$$I_1 := \int_1^n \frac{1}{x^p} dx = \left[-\frac{1}{p-1} \cdot \frac{1}{x^{p-1}} \right]_{x=1}^{x=n} = \frac{1}{1-p} \cdot \left(\frac{1}{n^{p-1}} - 1 \right)$$

$$I_2 := \int_1^n \frac{1}{x^{p+1}} dx = \left[-\frac{1}{p} \cdot \frac{1}{x^p} \right]_{x=1}^{x=n} = -\frac{1}{p} \left(\frac{1}{n^p} - 1 \right)$$

Hence

$$\int_1^n \frac{1}{x^p} dx - p \int_1^n \frac{1}{x^{p+1}} (x - \frac{1}{2}) dx = (1 - p) \cdot I_1 + \frac{p}{2} \cdot I_2 = \frac{1}{n^{p-1}} - \frac{1}{2} \cdot \frac{1}{n^p} - \frac{1}{2}.$$

Finally, combining everything gives

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{k^p} &= \left(\frac{1}{n^{p-1}} - \frac{1}{2} \cdot \frac{1}{n^p} - \frac{1}{2} \right) + p \int_1^n \frac{[x]}{x^{p+1}} dx + \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{n^p} \\ &= \frac{1}{n^{p-1}} + p \int_1^n \frac{[x]}{x^{p+1}} dx \end{aligned}$$

Comment. For (i), the basic idea is **integration by part**:

$$\int_a^b x f'(x) dx = \left[x f(x) \right]_{x=a}^{x=b} - \int_a^b f(x) dx$$

However, a rigorous proof is required as it did not appear in the lecture nor the tutorial. The key is to make use of the product rule and the Fundamental Theorem of Calculus. For (ii), we need to specify that the function $f(x) = 1/x^p$ is $\mathcal{C}^1$ on $(0, \infty)$ before applying the previous part. Also, the case $n = 1$ is not included and has to be checked separately. Finally, the case $p = 0$ needs special attention. Note that

$$\int_1^n \frac{1}{x^{p+1}} dx = \left[\ln(x) \right]_{x=1}^{x=n} \text{ if } p = 0; \quad \int_1^n \frac{1}{x^{p+1}} dx = \left[\frac{1}{p} \cdot \frac{1}{x^p} \right]_{x=1}^{x=n} \text{ if } p \neq 0, 1.$$

2. (15 points)

Let (f_n) be a sequence of bounded functions defined on $\mathbb{R}$. Suppose that $f(x) := \lim f_n(x)$ exists for all $x \in \mathbb{R}$.

(i). Show that

$$\lim_n \frac{f_1(x) + \cdots + f_n(x)}{n} = f(x)$$

for all $x \in \mathbb{R}$.

(ii). If we further assume that (f_n) converges uniformly to f on $\mathbb{R}$, does it imply that the sequence $(\frac{f_1 + \cdots + f_n}{n})$ converges uniformly to f on $\mathbb{R}$?

Solution.

(i.) We first show a weaker result by considering converging sequences of real numbers that converge to 0, that is, we first show that if (x_n) is a sequence of real numbers such that $\lim_n x_n = 0$. Then $c_n := \frac{1}{n}\sum_{i=1}^n x_i$ converges to 0.

Let $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that $|x_i| < \epsilon$ for all $n \geq N_1$. Furthermore by the Archimedean Principle, there exists $N_2 \in \mathbb{N}$ such that $\frac{1}{N_2} < \epsilon / \max\{1, \sum_{i=1}^{N_1} |x_i|\}$.
Next, we take some $N \in \mathbb{N}$ with $N \geq N_1, N_2$. Then for all $n \geq N$, we have

$$\left|\frac{1}{n}\sum_{i=1}^n x_i - 0\right| = \left|\frac{1}{n}\left(\sum_{i=1}^{N_1} x_i + \sum_{i=N_1+1}^{n} x_i\right)\right| \leq \underbrace{\frac{1}{n}\sum_{i=1}^{N_1}|x_i|}_{:=(I)} + \underbrace{\frac{1}{n}\sum_{i=N_1+1}^{n}|x_i|}_{:=(II)}$$

Then by the choice of N_1, N_2, we have that

$$(I) := \frac{1}{n}\sum_{i=1}^{N_1}|x_i| \leq \frac{1}{N_2}\sum_{i=1}^{N_1}|x_i| \leq \frac{1}{N_2}\max\{1, \sum_{i=1}^{N_1}|x_i|\} < \epsilon$$

$$(II) := \frac{1}{n}\sum_{i=N_1+1}^{n}|x_i| \leq \frac{1}{n}\sum_{i=N_1+1}^{n}\epsilon = \frac{n-N_1}{n}\epsilon \leq \epsilon$$

It then follows that

$$\left|\frac{1}{n}\sum_{i=1}^n x_i - 0\right| \leq (I) + (II) \leq 2\epsilon$$

for all $n \geq N$ (where the values of (I), (II) depends on n). Hence, by definition, we have $\lim_n \frac{1}{n}\sum_{i=1}^n x_i = 0$.

With a bit more effort we can strengthen the result to the case in question. Let (f_n) be a sequence of (bounded) functions on $\mathbb{R}$ such that $f_n \to f$ pointwise to some function f on $\mathbb{R}$. Then we fix $x \in \mathbb{R}$ and define $y_n : f_n(x) - f(x)$ for all $n \in \mathbb{N}$. Then it is clear that (y_n) is a sequence of real numbers converging to 0. Note that for all $n \in \mathbb{N}$, we have

$$\frac{1}{n}\sum_{i=1}^n y_i = \frac{1}{n}\sum_{i=1}^n f_i(x) - \frac{1}{n}\sum_{i=1}^n f(x) = \frac{1}{n}\sum_{i=1}^n f_i(x) - f(x)$$

By the weaker case it follows that $\lim_n \frac{1}{n}\sum_{i=1}^n f_i(x) - f(x) = 0$ and so $\lim_n \frac{1}{n}\sum_{i=1}^n f_i(x) = f(x)$ by the sum law of limit for all $x \in \mathbb{R}$.

(ii.) Yes. The form of the proof is exactly the same as that in part (a). In the following, we write $\|g\|_\infty := \sup_{x\in\mathbb{R}} |g(x)|$ whenever g is a bounded function.
First we leave it to the readers to show that $\|\cdot\|_\infty$ has the following properties:

(a) For all $\lambda \in \mathbb{R}$ and g bounded, we have $\|\lambda g\|_\infty = |\lambda| \|g\|_\infty$. *(Scalar Homogeneity)*

(b) For all g_1, g_2 bounded, we have $\|g_1 + g_2\|_\infty \leq \|g_1\|_\infty + \|g_2\|_\infty$ (which implies that $g_1 + g_2$ is bounded as well). *(Triangle Inequality)*

Then we proceed to the proof of the question.

Let $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that $\|f_n\|_\infty = \|f_n - 0\|_\infty < \epsilon$ for all $n \geq N_1$ since (f_n) converges uniformly to 0. Furthermore by the Archimedean Principle, there exists $N_2 \in \mathbb{N}$ such that $\frac{1}{N_2} < \epsilon / \max\{1, \sum_{i=1}^{N_1} \|f_i\|_\infty\}$.
Next, we take some $N \in \mathbb{N}$ with $N \geq N_1, N_2$. Then for all $n \geq N$, we have

$$\left\| \frac{1}{n} \sum_{i=1}^{n} f_i - 0 \right\|_\infty = \left\| \frac{1}{n} \left(\sum_{i=1}^{N_1} f_i + \sum_{i=N_1+1}^{n} f_i \right) \right\|_\infty \leq \underbrace{\frac{1}{n} \sum_{i=1}^{N_1} \|f_i\|_\infty}_{:=(I)} + \underbrace{\frac{1}{n} \sum_{i=N_1+1}^{n} \|f_i\|_\infty}_{:=(II)}$$

by both the scalar homogeneity and triangle inequality of $\|\cdot\|_\infty$. Then by the choice of N_1, N_2, we have that

$$(I) := \frac{1}{n} \sum_{i=1}^{N_1} \|f_i\|_\infty \leq \frac{1}{N_2} \sum_{i=1}^{N_1} \|f_i\|_\infty \leq \frac{1}{N_2} \max\{1, \sum_{i=1}^{N_1} \|f_i\|_\infty\} < \epsilon$$

$$(II) := \frac{1}{n} \sum_{i=N_1+1}^{n} \|f_i\|_\infty \leq \frac{1}{n} \sum_{i=N_1+1}^{n} \epsilon = \frac{n - N_1}{n} \epsilon \leq \epsilon$$

It then follows that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} f_i - 0 \right\|_\infty \leq (I) + (II) \leq 2\epsilon$$

for all $n \geq N$ (where the values of (I), (II) depends on n). Hence, by definition, we have $\lim_n \left\| \frac{1}{n} \sum_{i=1}^{n} f_i \right\|_\infty = 0$, which follows that $\frac{1}{n} \sum_{i=1}^{n} f_i$ converges uniformly to 0.

For the general case, suppose (f_n) is a sequence of bounded function converging to f uniformly. Then $\lim_n \|f_n - f\|_\infty = 0$. Define $g_n : f_n - f$ for all $n \in \mathbb{N}$. Since f is in fact a bounded function as it is the uniform limit for a sequence of bounded functions (see the proof in Question 3, Assignment 6), it follows from the triangle inequality of $\|\cdot\|_\infty$ that (g_n) is a sequence of bounded functions with $\lim_n \|g_n\|_\infty = 0$, that is, (g_n) converges to 0 uniformly. Note that for all $n \in \mathbb{N}$, we have

$$\frac{1}{n} \sum_{i=1}^{n} g_i = \frac{1}{n} \sum_{i=1}^{n} f_i - \frac{1}{n} \sum_{i=1}^{n} f = \frac{1}{n} \sum_{i=1}^{n} f_i - f$$

By the weaker case it follows that $\lim_n \left\| \frac{1}{n} \sum_{i=1}^{n} f_i - f \right\|_\infty = \lim_n \left\| \frac{1}{n} \sum_{i=1}^{n} g_i \right\|_\infty = 0$ and so $(\sum_{i=1}^{n} f_i)$ converges to f uniformly on $\mathbb{R}$.

Comment.

1. Let (x_n) be a sequence of real numbers. Then we call (x_n) to be Cesaro summable if $(\frac{1}{n}\sum_{i=1}^{n} x_i)$ converges. Part (i) shows that convergence of a sequence is stronger than the notion of Cesaro summable. In fact the latter is strictly weaker by considering the alternating sequence $((-1)^n)$, which is Cesaro summable but not converging.

2. The idea of the proof(s) is to truncate the Cesaro sum to two parts where the first (large) part can be regulated by the denominator n while the later part is small due to the convergence of the sequence.

3. For Part (i), in the regulation of the first large part, $\sum_{i=1}^{N_1} |x_i|$, we chose $N_2 \in \mathbb{N}$ such that $\frac{1}{N_2} < \epsilon / \max\{1, \sum_{i=1}^{N_1} |x_i|\}$. Alternatively, one may choose $N_2 \in \mathbb{N}$ such that we have instead

$$\frac{1}{N_2} < \epsilon / \max\{1, N_1 \cdot \sup_{1 \le i \le N_1} |x_i|\}$$

   The same is true for part (ii) with $\|f_i\|_\infty$ replacing $|x_i|$

4. We should emphasize that the two proofs are essentially the same. This is due to the similarity between $|\cdot|$ for real numbers and $\|\cdot\|_\infty$ for bounded functions. In fact, the proof is valid in any *normed space.*

5. Following (4), nonetheless, one should give the proof details for **BOTH** parts instead of simplying writing that the proofs are similar. Otherwise, marks would be deducted. After all, this is an *assessment* and we expect you to show to us your concrete understanding on the subject.

3. (20 points)

Let f be a continuous function defined on $[a,b]$. Assume that the right derivative of f exists for every $x \in (a,b)$, that is, the limit $f'_+(x) := \lim_{t\to 0+} \frac{f(x+t)-f(x)}{t}$ exists.

(i) If $f(b) < f(a)$, we define a function $h : (f(b), f(a)) \to \mathbb{R}$ by

$$h(y) := \sup\{x \in (a,b) : f(x) > y\}.$$

Show that $f(h(y)) = y$ for all $y \in (f(b), f(a))$.

(ii) Let $D := \{x \in (a,b) : f'_+(x) > 0\}$. Show that if $(a,b) \setminus D$ is countable, then f is increasing.

Solution.

i. First we show that h is well defined. For all $y \in (f(b), f(a))$, write $E_y := \{x \in (a,b) : f(x) > y\}$. Then $h(y) = \sup E_y$ by definition. We need to show that E_y is non-empty for all $y \in (f(b), f(a))$. This is due to the Intermediate Value Theorem: fix $y \in (f(b), f(a))$. Then we pick some $z \in (y, f(a)) \subset (f(b), f(a))$. Then there exists $x_z \in (a,b)$ such that $f(x_z) = z > y$ since f is continuous on $[a,b]$. It follows that $z \in E_y$ and so E_y is non-empty and h is well-defined by the Axiom of Completeness.

Next, we leave it an exercise that $h(y) \in [a,b]$ so $f(h(y))$ makes sense for any $y \in (f(b), f(a))$. (In fact the proof is similar to what comes next).

Then, we show that $f(h(y)) \geq y$ for all $y \in (f(b), f(a))$.
Fix $y \in (f(b), f(a))$. By the definition of supremum, it follows that there exists a sequence (x_n) in E_y such that $\lim_n x_n = \sup E_y = h(y)$. Since $x_n \in E_y$ for all $n \in \mathbb{N}$, $f(x_n) > y$ for all $n \in \mathbb{N}$. It follows by continuity of f at $h(y) \in [a,b]$ that $f(h(y)) = \lim_n f(x_n) \geq y$.

Finally, we show that $f(h(y)) \leq y$. Suppose not. We have $f(h(y)) > y$. Note that $h(y) \neq b$ otherwise $f(h(y)) = f(b) < y$ which contradicts to the last paragraph. Hence, $h(y) \in [a,b)$. Since $f(h(y)) > y$, by continuity of f at $h(y)$, it follows that there exists $0 < r$ such that $f(z) > y$ on $(h(y)-r, h(y)+r) \cap [a,b]$. By taking some $z \in [a,b]$ such that $0 < z - h(y) < r$ and $0 < z - h(y) < b - h(y)$ (why is it possible?), it follows that $z > h(y)$ and $z \in E_y$, which contradicts to the fact that $h(y) = \sup E_y$. Hence it must follow that $f(h(y)) \leq y$ and so $f(h(y)) = y$ together with $f(h(y)) \geq y$ for all $y \in (f(b), f(a))$.

ii. We proceed to prove its contrapositive. Suppose f is not increasing. Then there exists $s < t$ where $s, t \in [a,b]$ such that $f(s) > f(t)$. WLOG, we can assume $s, t \in (a,b)$ by continuity of f on $[a,b]$ (why?). Let $h : (f(t), f(s)) \to \mathbb{R}$ be defined by $h(y) := \{x \in (s,t) : f(x) > y\}$ for all $y \in (f(t), f(s))$. It is clear that f is continuous on $[s,t]$ (with right derivatives existing on (s,t), which was not used in the proof of part (i)). Hence, by part (i), it follows that h is well-defined and $f(h(y)) = y$ for all $y \in (f(t), f(s))$.

First, we show that $f'_+(h(y)) \leq 0$ and so $h(y) \notin D$ for all $y \in (f(t), f(s))$.
Fix $y \in (f(t), f(s))$. Then by the definition of supremum, we have $f(z) \leq y$ for all $z > h(y)$ and $z \in (s,t)$ and so for all $z \in (h(y), t)$, which is non-empty as $h(y) < t$ (why?)).
Hence, for all $z \in (h(y), t)$, we have

$$\frac{f(z) - f(h(y))}{z - h(y)} = \frac{f(z) - y}{z - h(y)} \leq 0$$

as $f(z) \leq y$. It follows that

$$f'_+(h(y)) := \lim_{z \to h(y)^+} \frac{f(z) - f(h(y))}{z - h(y)} \leq 0$$

and so $h(y) \notin D$ for all $y \in (f(t), f(s))$.

Next, we show that $\{h(y)\}_{y\in(f(t),f(s))}$ contains distinct elements, that is, h is injective. Suppose $h(y_1) = h(y_2)$ for some $y_1, y_2 \in (f(t), f(s))$. Then by part (i), we have that $y_1 = f(h(y_1)) = f(h(y_2)) = y_2$. By definition, h is injective.

Lastly, we show that $\{h(y)\}_{y\in(f(t),f(s))}$ is uncountable and so $(a, b)\backslash D$ is uncountable, which is what we want from the contrapositive argument. This follows from the fact that $(f(s), f(t))$ is an interval and so is of uncountably element by the Cantor Diagonal Theorem. By considering the injective map $h : (f(t), f(s)) \to [s, t]\backslash D \subset (a, b)\backslash D$, it follows that $\{h(y)\}_{y\in(f(t),f(s))} = h((f(t), f(s)))$ is uncountable and so its superset $(a, b)\backslash D$ is of uncountably many element .

Comment.

1. The proof of Q3i does NOT require the existence of left derivatives for f; only the continuity assumption of f has been used.

2. A set is uncountable if and only if it is not countable. Unlike the subsets $\mathbb{Q}$ and $\mathbb{Q}^c$ of $\mathbb{R}$, the uncountability of the *complement* of a set does not imply the countability of the set in general. An easy example is to consider an interval (a, b) and its complement in $\mathbb{R}$: both of the subsets are uncountable.