mmat5390: mathematical image processing

Assignment 1 solutions

1. (a) i. Note that H is a 4×4 matrix; hence it represents a linear transformation on 2×2 images.
H is not block-circulant. For example, consider the $y = 1, \beta = 1$-submatrix of H, i.e. $\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$. This is not a circulant matrix, as the shift-operator T maps $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ instead of $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Hence h is not shift-invariant with h_s being 2-periodic in both arguments.
(However, H is block-Toeplitz and thus h is shift-invariant.)
H is not a Kronecker product of two 2×2 matrices. For example, consider the $y = 1, \beta = 1$- and $y = 2, \beta = 1$-submatrices of H, i.e. $\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$. Neither is a scalar multiple of the other. Hence h is not separable.

ii. Note that H is a 9×9 matrix; hence it represents a linear transformation on 3×3 images.
H is not block-circulant. For example, consider the $y = 1, \beta = 2$-submatrix of h, i.e. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 0 & 0 \end{pmatrix}$, which is not a circulant matrix, as the shift-operator t maps $\begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$ to $\begin{pmatrix} 6 \\ 1 \\ 4 \end{pmatrix}$ instead of $\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$. Hence h is not shift-invariant with h_s being 3-periodic in both arguments. (Neither is H block-toeplitz, hence neither is h shift-invariant.)
H is the kronecker product of two 3×3 matrices; explicitly,

$$H = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 4 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 0 & 0 \end{pmatrix}.$$

Hence h is separable.

(b) i. Let $s = \alpha - x, t = \beta - y$. Then, $H(x, \alpha, y, \beta) = st + s^2$. Hence, H is shift-invariant.
Suppose H is separable. Then, there exists h_c, h_r such that $H(x, \alpha, y, \beta) = h_c(x, \alpha)h_r(y, \beta)$.
We can then deduce the following results:
$H(1, 2, 2, 3) = h_c(1.2)h_r(2, 3) = 2$ and $H(1, 2, 2, 4) = h_c(1, 2)h_r(2, 3) = 3$
$\implies h_r(2, 3)/h_r(2, 4) = 2/3$.
But, $H(3, 2, 2, 3) = h_c(1, 2)h_r(2, 3) = 0$ and $H(3, 2, 2, 4) = h_c(1, 2)h_r(2, 4) = -1$
$\implies h_r(2, 3)/h_r(2, 4) = 0 \neq 2/3$.
Hence, H is not separable.

ii. Note that $H(1, 2, 1, 1) = \frac{2}{7e}$, $H(2, 3, 1, 1) = \frac{3}{7e^4}$, $H(1, 2, 1, 1) \neq H(2, 3, 1, 1)$, hence H is not shift-invariant.
Let $g_1(x, \alpha) = \alpha e^{-x^2}$ and $g_2(y, \beta) = \beta^2 + y^2 + 5$, then $h(x, \alpha, y, \beta) = g_1(x, \alpha)g_2(y, \beta)$, hence it is separable.

2. (a) Let h be the shift-invariant PSF of a linear image transformation on $M_{n\times n}(\mathbb{R})$, with h_s n-periodic in both arguments such that $h(x, \alpha, y, \beta) = h_s(\alpha - x, \beta - y)$. Let H be the corresponding transformation matrix.

Let $a \in \mathbb{Z}$. Then

$$\begin{aligned} H\left(\alpha+(\beta+a-1)n, x+(y+a-1)n\right) &= h(x,\alpha,y+a,\beta+a) \\ &= h_s\left(\alpha-x,(\beta+a)-(y+a)\right) \\ &= h_s(\alpha-x,\beta-y) \\ &= h(x,\alpha,y,\beta) \\ &= H\left(\alpha+(\beta-1)n, x+(y-1)n\right). \end{aligned}$$

Also, by periodicity of h_s, for $y \in \mathbb{N} \cap [1, n-1]$,

$$\begin{aligned} H(\alpha+(n-1)n, x+(y-1)n) &= h(x,\alpha,y,n) \\ &= h_s(\alpha-x,n-y) \\ &= h_s(\alpha-x,1-(y+1) \\ &= h(x,\alpha,y+1,1) \\ &= H(\alpha,x+yn) \end{aligned}$$

and for $\beta \in \mathbb{N} \cap [1, n-1]$,

$$\begin{aligned} H(\alpha+(\beta-1)n, x+(n-1)n) &= h(x,\alpha,n,\beta) \\ &= h_s(\alpha-x,\beta-n) \\ &= h_s(\alpha-x,(\beta+1)-1) \\ &= h(x,\alpha,1,\beta+1) \\ &= H(\alpha+\beta n,x). \end{aligned}$$

Hence H is circulant when viewed as a matrix consisting of blocks of fixed (y, β)-values. Combined with the result of Theorem 1.13, we establish that H is block-circulant.

(b) Let h be the shift-invariant PSF of a linear image transformation on $M_{n\times n}(\mathbb{R})$ in the sense that $h(x,\alpha,y,\beta) = h_s(\alpha-x,\beta-y)$. Let H be the corresponding transformation matrix.
Fix y and β. Then for any α, x and $a \in \mathbb{N}$ satisfying $a \leq n - \max\{\alpha, x\}$,

$$\begin{aligned} h\left(x+an,\alpha+an,y,\beta\right) &= h_s\left(\alpha+an-x-an,\beta-y\right) \\ &= h_s(\alpha-x,\beta-y) \\ &= h(x,\alpha,y,\beta) \end{aligned}$$

On the other hand, fix x and α. Then for any β, y and $a \in \mathbb{N}$ satisfying $a \leq n - \max\{\beta, y\}$,

$$\begin{aligned} h(\alpha,x,\beta+an,y+an) &= h_s(\alpha-x,\beta+an-y-an) \\ &= h_s(\alpha-x,\beta-y) \\ &= h(x,\alpha,y,\beta) \end{aligned}$$

Hence, we know H is block Toeplitz.
Reverse all the statements shown above, we know h is shift-invariant if H is block Toeplitz.

3. (a) For simplicity (and to guide the indexing of $f * g$), we only consider the cases where f and g are indexed with the same set of indices.
If f and g are indexed with $1 \leq i, j \leq 2$, i.e. if

$$f = (f(i,j))_{1\leq i,j\leq 2} = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} \text{ and } g = (g(i,j))_{1\leq i,j\leq 2} = \begin{pmatrix} 1 & 4 \\ 0 & 6 \end{pmatrix},$$

then

$$
\begin{aligned}
f * g(1,1) &= f(1,1)g(2,2) + f(1,2)g(2,1) + f(2,1)g(1,2) + f(2,2)g(1,1) \\
&= 6 \cdot 6 + 3 \cdot 0 + 3 \cdot 4 + 2 \cdot 1 = 50, \\
f * g(1,2) &= f(1,1)g(2,1) + f(1,2)g(2,2) + f(2,1)g(1,1) + f(2,2)g(1,2) \\
&= 6 \cdot 0 + 3 \cdot 6 + 3 \cdot 1 + 2 \cdot 4 = 29, \\
f * g(2,1) &= f(1,1)g(1,2) + f(1,2)g(1,1) + f(2,1)g(2,2) + f(2,2)g(2,1) \\
&= 6 \cdot 4 + 3 \cdot 1 + 3 \cdot 6 + 2 \cdot 0 = 45, \text{ and} \\
f * g(2,2) &= f(1,1)g(1,1) + f(1,2)g(1,2) + f(2,1)g(2,1) + f(2,2)g(2,2) \\
&= 6 \cdot 1 + 3 \cdot 4 + 3 \cdot 0 + 2 \cdot 6 = 30,
\end{aligned}
$$

i.e. $f * g = (f * g(i,j))_{1 \le i,j \le 2} = \begin{pmatrix} 50 & 29 \\ 45 & 30 \end{pmatrix}$.

$$
\begin{aligned}
g * f(1,1) &= g(1,1)f(2,2) + g(1,2)f(2,1) + g(2,1)f(1,2) + g(2,2)f(1,1) \\
&= 1 \cdot 2 + 4 \cdot 3 + 0 \cdot 3 + 6 \cdot 6 = 50, \\
g * f(1,2) &= g(1,1)f(2,1) + g(1,2)f(2,2) + g(2,1)f(1,1) + g(2,2)f(1,2) \\
&= 1 \cdot 3 + 4 \cdot 2 + 0 \cdot 6 + 6 \cdot 3 = 29, \\
g * f(2,1) &= g(1,1)f(1,2) + g(1,2)f(1,1) + g(2,1)f(2,2) + g(2,2)f(2,1) \\
&= 1 \cdot 3 + 4 \cdot 6 + 0 \cdot 2 + 6 \cdot 3 = 45, \text{ and} \\
g * f(2,2) &= g(1,1)f(1,1) + g(1,2)f(1,2) + g(2,1)f(2,1) + g(2,2)f(2,2) \\
&= 1 \cdot 6 + 4 \cdot 3 + 0 \cdot 3 + 6 \cdot 2 = 30,
\end{aligned}
$$

i.e. $g * f = (g * f(i,j))_{1 \le i,j \le 2} = \begin{pmatrix} 50 & 29 \\ 45 & 30 \end{pmatrix}$.

(b) Let $f, g \in M_{m \times n}(\mathbb{R})$, and assume that they are periodically extended. Let $\alpha \in \mathbb{N} \cap [1, m]$ and $\beta \in \mathbb{N} \cap [1, n]$. By definition,

$$
\begin{aligned}
f * g(\alpha, \beta) &= \sum_{x=1}^{m} \sum_{y=1}^{n} f(x,y) g(\alpha - x, \beta - y) \\
&= \sum_{i=\alpha-m}^{\alpha-1} \sum_{j=\beta-n}^{\beta-1} f(\alpha - i, \beta - j) g(i,j) \text{ (letting } i = \alpha - x,\ j = \beta - y) \\
&= \sum_{i=\alpha-m}^{0} \sum_{j=\beta-n}^{0} f(\alpha - i, \beta - j) g(i,j) + \sum_{i=\alpha-m}^{0} \sum_{j=1}^{\beta-1} f(\alpha - i, \beta - j) g(i,j) \\
&\quad + \sum_{i=1}^{\alpha-1} \sum_{j=\beta-n}^{0} f(\alpha - i, \beta - j) g(i,j) + \sum_{i=1}^{\alpha-1} \sum_{j=1}^{\beta-1} f(\alpha - i, \beta - j) g(i,j) \\
&= \sum_{i=\alpha}^{m} \sum_{j=\beta}^{n} f(\alpha - i, \beta - j) g(i,j) + \sum_{i=\alpha}^{m} \sum_{j=1}^{\beta-1} f(\alpha - i, \beta - j) g(i,j) \\
&\quad + \sum_{i=1}^{\alpha-1} \sum_{j=\beta}^{n} f(\alpha - i, \beta - j) g(i,j) + \sum_{i=1}^{\alpha-1} \sum_{j=1}^{\beta-1} f(\alpha - i, \beta - j) g(i,j) \text{ (by periodicity)} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} g(i,j) f(\alpha - i, \beta - j) \\
&= g * f(\alpha, \beta);
\end{aligned}
$$

hence $f * g = g * f$.

4. (a) We start by finding the eigenvalues and corresponding orthonormal eigenbasis of $A^T A$.

$$
A^T A = \begin{pmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 16 \end{pmatrix}
$$

$p(\lambda) = (det)(A^TA - \lambda I_3) = (5-\lambda)^2(16-\lambda) - 16(16-\lambda) = -(\lambda-1)(\lambda-9)(\lambda-16)$.
So, the eigenvalues of A^TA are $\lambda_1 = 16, \lambda_2 = 9, \lambda_3 = 1$.

The corresponding eigenvectors are $v_1 = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$, and $v_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$.

Then we compute the matrix U. $u_1 = \frac{1}{\sigma_1}Av_1 = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $u_2 = \frac{1}{\sigma_2}Av_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$, and

$u_3 = \frac{1}{\sigma_3}Av_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$

So, $A = U\Sigma V^T = \begin{pmatrix}0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}4&0&0\\0&3&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0\end{pmatrix}$

(b) We can then write A as $A = 4u_1v_1^T + 3u_2v_2^T + 1u_3v_3^T = 4\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} + 3\begin{pmatrix}1/2&1/2&0\\1/2&1/2&0\\0&0&0\end{pmatrix} +$
$\begin{pmatrix}1/2&-1/2&0\\-1/2&1/2&0\\0&0&0\end{pmatrix}$

5. (a) Note that
$h(1,1,1,1) = 2,\ h(2,1,1,1) = 0,\ h(1,1,2,1) = 8,\ h(2,1,2,1) = 0,$
$h(1,2,1,1) = 1,\ h(2,2,1,1) = 2,\ h(1,2,2,1) = 4,\ h(2,2,2,1) = 8,$
$h(1,1,1,2) = 6,\ h(2,1,1,2) = 0,\ h(1,1,2,2) = 4,\ h(2,1,2,2) = 0,$
$h(1,2,1,2) = 3,\ h(2,2,1,2) = 6,\ h(1,2,2,2) = 2,\ h(2,2,2,2) = 4.$
Define $g_1 : \{1,2\}^2 \to \mathbb{R}$ by $g_1(1,1) = 2,\ g_1(2,1) = 0,\ g_1(1,2) = 1,\ g_1(2,2) = 2.$
and $g_2 : \{1,2\}^2 \to \mathbb{R}$ by $g_2(1,1) = 1,\ g_2(2,1) = 4,\ g_2(1,2) = 3,\ g_2(2,2) = 2$
As $h(x,\alpha,y,\beta) = g_1(x,\alpha)g_2(y,\beta)$ for all $1 \le x,\alpha,y,\beta) \le 2$, h is separable.
And we observe that

$$H = \begin{pmatrix}2&0&8&0\\1&2&4&8\\6&0&4&0\\3&6&2&4\end{pmatrix} = \begin{pmatrix}1&4\\3&2\end{pmatrix} \otimes \begin{pmatrix}2&0\\1&2\end{pmatrix}.$$

(b) Let h be the separable PSF of a linear image transformation, with $h(x,\alpha,y,\beta) = h_c(x,\alpha)h_r(y,\beta)$. Let H be the corresponding transformation matrix.
Then the $y = k, \beta = l$-submatrix of H (denoted by $\tilde{H}_{kl}$) is given by

$$\left(\alpha\downarrow \begin{matrix} x\rightarrow \\ \begin{pmatrix} y=k \\ \beta = l\end{pmatrix}\end{matrix} \right) = [H(\alpha + (l-1)n, x + (k-1)n)]_{\substack{1\le x\le n\\ 1\le\alpha\le n}}$$
$$= [h(x,\alpha,k,l)]_{\substack{1\le x\le n\\ 1\le\alpha\le n}}$$
$$= [h_c(x,\alpha)h_r(k,l)]_{\substack{1\le x\le n\\ 1\le\alpha\le n}}$$
$$= h_r(k,l)[h_c(x,\alpha)]_{\substack{1\le x\le n\\ 1\le\alpha\le n}}$$
$$= h_r(k,l)h_c^T.$$

Recall that

$$
H = \begin{pmatrix}
\begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix} \end{pmatrix} &
\begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=1 \end{pmatrix} \end{pmatrix} & \cdots &
\begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=n \\ \beta=1 \end{pmatrix} \end{pmatrix} \\
\begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=2 \end{pmatrix} \end{pmatrix} &
\begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=2 \end{pmatrix} \end{pmatrix} & \cdots &
\begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=n \\ \beta=2 \end{pmatrix} \end{pmatrix} \\
\vdots & \vdots & \ddots & \vdots \\
\begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=n \end{pmatrix} \end{pmatrix} &
\begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=n \end{pmatrix} \end{pmatrix} & \cdots &
\begin{pmatrix} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=n \\ \beta=n \end{pmatrix} \end{pmatrix}
\end{pmatrix}
$$

$$
= \begin{pmatrix} \tilde{H}_{11} & \tilde{H}_{21} & \cdots & \tilde{H}_{n1} \\ \tilde{H}_{12} & \tilde{H}_{22} & \cdots & \tilde{H}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{H}_{1n} & \tilde{H}_{2n} & \cdots & \tilde{H}_{nn} \end{pmatrix} = \begin{pmatrix} h_r(1,1)h_c^T & h_r(2,1)h_c^T & \cdots & h_r(n,1)h_c^T \\ h_r(1,2)h_c^T & h_r(2,2)h_c^T & \cdots & h_r(n,2)h_c^T \\ \vdots & \vdots & \ddots & \vdots \\ h_r(1,n)h_c^T & h_r(2,n)h_c^T & \cdots & h_r(n,n)h_c^T \end{pmatrix}
$$

$$
= \begin{pmatrix} h_r^T(1,1)h_c^T & h_r^T(1,2)h_c^T & \cdots & h_r^T(1,n)h_c^T \\ h_r^T(2,1)h_c^T & h_r^T(2,2)h_c^T & \cdots & h_r^T(2,n)h_c^T \\ \vdots & \vdots & \ddots & \vdots \\ h_r^T(n,1)h_c^T & h_r^T(n,2)h_c^T & \cdots & h_r^T(n,n)h_c^T \end{pmatrix} = h_r^T \otimes h_c^T.
$$