MMAT5390: Mathematical Image Processing

Practice Final solutions

1. Please refer to the Chapter Solutions.

2.
$$\begin{cases} 3 & = h * I(1,1) = \frac{1}{4}[I(0,0) + I(1,0) + I(0,1) + I(1,1)] = a + b + c \\ -4 & = I_1(1,2) = 2I(1,2) - I(0,2) - I(1,1) = 0 - c - (2a + c) = -2(a + c) \\ \dfrac{1}{2} & = \dfrac{1}{1 + (\frac{a^2}{4})^b}. \end{cases}$$

   Hence
$$(\frac{a^2}{4})^b = 1 \implies \frac{a^2}{4} = 1 \text{ or } b = 0,$$
   where $\frac{a^2}{4} = 1$ implies:
$$a^2 = 4 \implies a = 2 \implies -2(2 + c) = -4 \implies c = 0 \implies b = 1,$$
   whereas $b = 0$ implies:
$$\begin{cases} a + c & = 3 \\ -2a - 2c & = -4 \end{cases},$$
   which has no solution.

   Hence $a = 2$, $b = 1$ and $c = 0$.

   P.S. The condition that $DFT(I)(2,0) \neq 0$ imposes the constraint $3b \neq c$, which is satisfied by $(a, b, c) = (2, 1, 0)$.

3. (a) Note that
$$G_x(f)(x,y) = h_x * f(x,y) \text{ and } G_y(f)(x,y) = h_y * f(x,y),$$
   where
$$h_x(x,y) = \begin{cases} \frac{1}{2} & \text{if } (x,y) = (0,0) \\ \frac{1}{4} & \text{if } (x,y) = (-1,0) \text{ or } (1,0) \\ 0 & \text{otherwise} \end{cases}$$
   and
$$h_y(x,y) = \begin{cases} \frac{1}{2} & \text{if } (x,y) = (0,0) \\ \frac{1}{4} & \text{if } (x,y) = (0,-1) \text{ or } (0,1) \\ 0 & \text{otherwise.} \end{cases}$$
   Hence $G_x(G_y(f)) = h_x * (h_y * f) = (h_x * h_y) * f = h * f$, where
$$\begin{aligned} h(x,y) &= h_x * h_y(x,y) \\ &= \begin{cases} \frac{1}{4} & \text{if } (x,y) = (0,0) \\ \frac{1}{8} & \text{if } (x,y) = (0,-1) \text{ or } (-1,0) \text{ or } (1,0) \text{ or } (0,1) \\ \frac{1}{16} & \text{if } (x,y) = (-1,-1) \text{ or } (1,-1) \text{ or } (-1,1) \text{ or } (1,1) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

   (b) Recall that $DFT(h * f)(u,v) = N^2 DFT(h)(u,v) DFT(f)(u,v)$.

Hence to perform unsharp masking on $f \in M_{4\times 4}$,

$$
\begin{aligned}
H &= 16DFT(h) \\
&= \frac{1}{16}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\begin{pmatrix}4 & 2 & 0 & 2\\ 2 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 2 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix}8 & 4 & 0 & 4\\ 4 & 2 & 0 & 2\\ 0 & 0 & 0 & 0\\ 4 & 2 & 0 & 2\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix}16 & 8 & 0 & 8\\ 8 & 4 & 0 & 4\\ 0 & 0 & 0 & 0\\ 8 & 4 & 0 & 4\end{pmatrix} = \frac{1}{4}\begin{pmatrix}4 & 2 & 0 & 2\\ 2 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 2 & 1 & 0 & 1\end{pmatrix}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
DFT(f) &= \frac{1}{16}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\begin{pmatrix}4 & 1 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix}6 & 1 & 0 & 1\\ 4 & 1 & 0 & 1\\ 2 & 1 & 0 & 1\\ 4 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix}8 & 6 & 4 & 6\\ 6 & 4 & 2 & 4\\ 4 & 2 & 0 & 2\\ 6 & 4 & 2 & 4\end{pmatrix} = \frac{1}{8}\begin{pmatrix}4 & 3 & 2 & 3\\ 3 & 2 & 1 & 2\\ 2 & 1 & 0 & 1\\ 3 & 2 & 1 & 2\end{pmatrix}.
\end{aligned}
$$

$\tilde{F}(u,v) = DFT(f)(u,v)[2 - H(u,v)]$ and thus

$$
\tilde{F} = \frac{1}{32}\begin{pmatrix}4 & 3 & 2 & 3\\ 3 & 2 & 1 & 2\\ 2 & 1 & 0 & 1\\ 3 & 2 & 1 & 2\end{pmatrix} \odot \begin{pmatrix}4 & 6 & 8 & 6\\ 6 & 7 & 8 & 7\\ 8 & 8 & 8 & 8\\ 6 & 7 & 8 & 7\end{pmatrix} = \frac{1}{32}\begin{pmatrix}16 & 18 & 16 & 18\\ 18 & 14 & 8 & 14\\ 16 & 8 & 0 & 8\\ 18 & 14 & 8 & 14\end{pmatrix} = \frac{1}{16}\begin{pmatrix}8 & 9 & 8 & 7\\ 9 & 7 & 4 & 7\\ 8 & 4 & 0 & 4\\ 9 & 7 & 4 & 7\end{pmatrix}
$$

$$
\begin{aligned}
\text{and } \tilde{f} &= \frac{1}{16}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{pmatrix}\begin{pmatrix}8 & 9 & 8 & 7\\ 9 & 7 & 4 & 7\\ 8 & 4 & 0 & 4\\ 9 & 7 & 4 & 7\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix}34 & 27 & 16 & 27\\ 0 & 5 & 8 & 5\\ -2 & -1 & 0 & -1\\ 0 & 5 & 8 & 5\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix}104 & 18 & -4 & 18\\ 18 & -8 & -2 & -8\\ -4 & -2 & 0 & -2\\ 18 & -8 & -2 & -8\end{pmatrix} \\
&= \frac{1}{8}\begin{pmatrix}52 & 9 & -2 & 9\\ 9 & -4 & -1 & -4\\ -2 & -1 & 0 & -1\\ 9 & -4 & -1 & -4\end{pmatrix}.
\end{aligned}
$$

4.

$$
\begin{aligned}
DFT(g)(u,v) &= \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} g(x,y)e^{-2\pi j\frac{ux+vy}{N}} \\
&= \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sum_{k=0}^{T} I(x-ck,y-ck)e^{-2\pi j\frac{ux+vy}{N}} \\
&= \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sum_{k=0}^{T} I(x-ck,y-ck)e^{-2\pi j\frac{u(x-ck)+v(y-ck)}{N}}e^{-2\pi j\frac{ck(u+v)}{N}} \\
&= \frac{1}{N^2}\sum_{k=0}^{T}\sum_{x'=-ck}^{N-ck-1}\sum_{y'=-ck}^{N-ck-1} I(x',y')e^{-2\pi j\frac{ux'+vy'}{N}}e^{-2\pi j\frac{ck(u+v)}{N}} \\
&= \left(\sum_{k=0}^{T} e^{-2\pi j\frac{ck(u+v)}{N}}\right)\left[\frac{1}{N^2}\sum_{x'=0}^{N-1}\sum_{y'=0}^{N-1} I(x',y')e^{-2\pi j\frac{ux'+vy'}{N}}\right] \\
&= \begin{cases} \frac{1-e^{-2\pi j\frac{c(T+1)(u+v)}{N}}}{1-e^{-2\pi j\frac{c(u+v)}{N}}}DFT(I)(u,v) & \text{if } c(u+v)\notin N\mathbb{Z}, \\ (T+1)DFT(I)(u,v) & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{e^{-\pi j\frac{c(T+1)(u+v)}{N}}\left(e^{\pi j\frac{c(T+1)(u+v)}{N}}-e^{-\pi j\frac{c(T+1)(u+v)}{N}}\right)}{e^{-\pi j\frac{c(u+v)}{N}}\left(e^{\pi j\frac{c(u+v)}{N}}-e^{-\pi j\frac{c(u+v)}{N}}\right)}DFT(I)(u,v) & \text{if } c(u+v)\notin N\mathbb{Z}, \\ (T+1)DFT(I)(u,v) & \text{otherwise,} \end{cases} \\
&= \begin{cases} e^{-\pi j\frac{cT(u+v)}{N}}\frac{\sin\frac{c\pi(T+1)(u+v)}{N}}{\sin\frac{c\pi(u+v)}{N}}DFT(I)(u,v) & \text{if } c(u+v)\notin N\mathbb{Z}, \\ (T+1)DFT(I)(u,v) & \text{otherwise.} \end{cases}
\end{aligned}
$$

Hence $H(u,v) = \begin{cases} e^{-\pi j\frac{cT(u+v)}{N}}\frac{\sin\frac{c\pi(T+1)(u+v)}{N}}{\sin\frac{c\pi(u+v)}{N}} & \text{if } c(u+v)\notin N\mathbb{Z}, \\ T+1 & \text{otherwise.} \end{cases}$

5. (a) Recall that the definition of Kronecker product is

$$
A\otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & \cdots & a_{2N}B \\ \vdots & \vdots & & \vdots \\ a_{N1}B & a_{N2}B & \cdots & a_{NN}B \end{pmatrix}
$$

We have

$$
W = W_N\otimes W_N = \begin{pmatrix} \frac{1}{\sqrt{N}}e^{2\pi j\frac{0\cdot 0}{N}}\cdot W_N & \frac{1}{\sqrt{N}}e^{2\pi j\frac{0\cdot 1}{N}}\cdot W_N & \cdots & \frac{1}{\sqrt{N}}e^{2\pi j\frac{0\cdot(N-1)}{N}}\cdot W_N \\ \frac{1}{\sqrt{N}}e^{2\pi j\frac{1\cdot 0}{N}}\cdot W_N & \frac{1}{\sqrt{N}}e^{2\pi j\frac{1\cdot 1}{N}}\cdot W_N & \cdots & \frac{1}{\sqrt{N}}e^{2\pi j\frac{1\cdot(N-1)}{N}}\cdot W_N \\ \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{N}}e^{2\pi j\frac{(N-1)\cdot 0}{N}}\cdot W_N & \frac{1}{\sqrt{N}}e^{2\pi j\frac{(N-1)\cdot 1}{N}}\cdot W_N & \cdots & \frac{1}{\sqrt{N}}e^{2\pi j\frac{(N-1)\cdot(N-1)}{N}}\cdot W_N \end{pmatrix}
$$

And rewrite it as $W = \left(\frac{1}{\sqrt{N}}e^{2\pi j\frac{n\cdot k}{N}}\cdot W_N\right)_{0\le n,k\le N-1}$

Since $\overline{W_N} = \left(\frac{1}{\sqrt{N}}e^{-2\pi j\frac{nk}{N}}\right)_{0\le n,k\le N-1}$, we know that

$$
\overline{W_N}\otimes\overline{W_N} = \begin{pmatrix} \frac{1}{\sqrt{N}}e^{-2\pi j\frac{0\cdot 0}{N}}\cdot \overline{W_N} & \frac{1}{\sqrt{N}}e^{-2\pi j\frac{0\cdot 1}{N}}\cdot \overline{W_N} & \cdots & \frac{1}{\sqrt{N}}e^{-2\pi j\frac{0\cdot(N-1)}{N}}\cdot \overline{W_N} \\ \frac{1}{\sqrt{N}}e^{-2\pi j\frac{1\cdot 0}{N}}\cdot \overline{W_N} & \frac{1}{\sqrt{N}}e^{-2\pi j\frac{1\cdot 1}{N}}\cdot \overline{W_N} & \cdots & \frac{1}{\sqrt{N}}e^{-2\pi j\frac{1\cdot(N-1)}{N}}\cdot \overline{W_N} \\ \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{N}}e^{-2\pi j\frac{(N-1)\cdot 0}{N}}\cdot \overline{W_N} & \frac{1}{\sqrt{N}}e^{-2\pi j\frac{(N-1)\cdot 1}{N}}\cdot \overline{W_N} & \cdots & \frac{1}{\sqrt{N}}e^{-2\pi j\frac{(N-1)\cdot(N-1)}{N}}\cdot \overline{W_N} \end{pmatrix}
$$

And rewrite it as $\overline{W_N} \otimes \overline{W_N} = \left(\frac{1}{\sqrt{N}} e^{-2\pi j \frac{n \cdot k}{N}} \cdot \overline{W_N} \right)_{0 \le n,k \le N-1}$

Using block-matrix multiplication, we can calculate

$$
\begin{aligned}
W \cdot (\overline{W_N} \otimes \overline{W_N}) &= \left(\frac{1}{\sqrt{N}} e^{2\pi j \frac{n \cdot k}{N}} \cdot W_N \right)_{0 \le n,k \le N-1} \left(\frac{1}{\sqrt{N}} e^{-2\pi j \frac{n \cdot k}{N}} \cdot \overline{W_N} \right)_{0 \le n,k \le N-1} \\
&= \left(\sum_{p=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi j \frac{n \cdot p}{N}} \cdot W_N \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{p \cdot k}{N}} \cdot \overline{W_N} \right)_{0 \le n,k \le N-1} \\
&= \left(\sum_{p=0}^{N-1} \frac{1}{N} e^{2\pi j \frac{(n-k) \cdot p}{N}} \cdot W_N \overline{W_N} \right)_{0 \le n,k \le N-1} \\
&= I_{N^2}
\end{aligned}
$$

Therefore, $W^{-1} = \overline{W_N} \otimes \overline{W_N}$.

(b) Note that $f = (f_{i,j})_{0 \le i,j \le N-1} \in M_{N \times N}(\mathbb{C})$ then

$$
\mathcal{S}(f) = \begin{pmatrix} f_{0,0} & f_{1,0} & \cdots & f_{N-1,0} & \cdots & f_{0,N-1} & f_{1,N-1} & \cdots & f_{N-1,N-1} \end{pmatrix}^T \in M_{N^2 \times 1}(\mathbb{C})
$$

From (a) we know that $W^{-1} = \overline{W_N} \otimes \overline{W_N} = \left(\frac{1}{\sqrt{N}} e^{-2\pi j \frac{n \cdot k}{N}} \cdot \overline{W_N} \right)_{0 \le n,k \le N-1} \in M_{N^2 \times N^2}(\mathbb{C})$

Then, $W^{-1}\mathcal{S}(f) \in M_{N^2 \times 1}(\mathbb{C})$, and its l-th entry is

$$
\begin{aligned}
\left(W^{-1}\mathcal{S}(f)\right)_l &= W^{-1}(l,:) \cdot \mathcal{S}(f) \\
&= \begin{pmatrix}
\frac{1}{\sqrt{N}} e^{-2\pi j \frac{\lfloor \frac{l}{N} \rfloor \cdot 0}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{\mathrm{mod}_N(l) \cdot 0}{N}} \\
\frac{1}{\sqrt{N}} e^{-2\pi j \frac{\lfloor \frac{l}{N} \rfloor \cdot 0}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{\mathrm{mod}_N(l) \cdot 1}{N}} \\
\vdots \\
\frac{1}{\sqrt{N}} e^{-2\pi j \frac{\lfloor \frac{l}{N} \rfloor \cdot 0}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{\mathrm{mod}_N(l) \cdot (N-1)}{N}} \\
\vdots \\
\frac{1}{\sqrt{N}} e^{-2\pi j \frac{\lfloor \frac{l}{N} \rfloor \cdot (N-1)}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{\mathrm{mod}_N(l) \cdot 0}{N}} \\
\frac{1}{\sqrt{N}} e^{-2\pi j \frac{\lfloor \frac{l}{N} \rfloor \cdot (N-1)}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{\mathrm{mod}_N(l) \cdot 1}{N}} \\
\vdots \\
\frac{1}{\sqrt{N}} e^{-2\pi j \frac{\lfloor \frac{l}{N} \rfloor \cdot (N-1)}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{\mathrm{mod}_N(l) \cdot (N-1)}{N}}
\end{pmatrix}^T
\begin{pmatrix} f_{0,0} \\ f_{1,0} \\ \vdots \\ f_{N-1,0} \\ \vdots \\ f_{0,N-1} \\ f_{1,N-1} \\ \vdots \\ f_{N-1,N-1} \end{pmatrix} \\
&= \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} \frac{1}{N} e^{-2\pi j \frac{\lfloor \frac{l}{N} \rfloor \cdot q + \mathrm{mod}_N(l) \cdot p}{N}} f_{p,q}
\end{aligned}
$$

From $\hat{f}_{p,q} = \frac{1}{N^2} \sum_{\alpha,\beta=0}^{N-1} f_{\alpha,\beta} \cdot e^{-2\pi j \frac{p\alpha + q\beta}{N}}$, we have $\mathcal{S}(\hat{f}) = \begin{pmatrix} \hat{f}_{0,0} \\ \hat{f}_{1,0} \\ \vdots \\ \hat{f}_{N-1,0} \\ \vdots \\ \hat{f}_{0,N-1} \\ \hat{f}_{1,N-1} \\ \vdots \\ \hat{f}_{N-1,N-1} \end{pmatrix} \in M_{N^2 \times 1}(\mathbb{C})$

and $(N \cdot (\mathcal{S}(f)))_l = N\hat{f}(\mathrm{mod}_N(l), \lfloor \frac{l}{N} \rfloor) = N \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} \frac{1}{N^2} e^{-2\pi j \frac{\lfloor \frac{l}{N} \rfloor \cdot q + \mathrm{mod}_N(l) \cdot p}{N}} f_{p,q}$

Therefore, $W^{-1}\mathcal{S}(f) = N\mathcal{S}(\hat{f})$ for any $f \in M_{N \times N}(\mathbb{C})$, where $\hat{f} = DFT(f)$.

6. (a) Since D is a real block-circulant matrix, let $D = \begin{pmatrix} D_0 & D_{N-1} & \cdots & D_1 \\ D_1 & D_0 & \cdots & D_2 \\ \cdots & \cdots & \cdots & \cdots \\ D_{N-1} & D_{N-2} & \cdots & D_0 \end{pmatrix}$, where D_n is a real circulant matrix.

Denote by $D_{n,m}$ the value of the entries on the diagonal of D_n with indices $\{(x,y) : x - y \in m + N\mathbb{Z}\}$.

i. First, we prove that $\overline{W_N} D_n W_N$ is a diagonalization of D_n.
For any $x, y \in \{0, 1, \cdots, N-1\}$,

$$\begin{aligned}
\overline{W_N} D_n W_N(x,y) &= \sum_{s=0}^{N-1} \overline{W_N}(x,s)[D_n W_N](s,y) \\
&= \sum_{s=0}^{N-1} \overline{W_N}(x,s) \sum_{t=0}^{N-1} D_n(s,t) W_N(t,y) \\
&= \frac{1}{N} \sum_{s=0}^{N-1} \sum_{t=0}^{N-1} D_{n,s-t} e^{-2\pi j \frac{sx-ty}{N}} \\
&= \frac{1}{N} \sum_{s=0}^{N-1} \sum_{t'=s-N+1}^{s} D_{n,t'} e^{-2\pi j \frac{sx-(s-t')y}{N}} \\
&= \frac{1}{N} \sum_{s=0}^{N-1} e^{-2\pi j \frac{s(x-y)}{N}} \sum_{t'=0}^{N-1} D_{n,t'} e^{-2\pi j \frac{t'y}{N}} \\
&= \begin{cases} \sum\limits_{m=0}^{N-1} D_{n,m} e^{-2\pi j \frac{my}{N}} & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

ii. Then we prove the following part.
For any $k, l \in \mathbb{Z} \cap [0, N-1]$,

$$\begin{aligned}
[W^{-1} D W]_{k,l} &= [(\overline{W_N} \otimes \overline{W_N}) D (W_N \otimes W_N)]_{k,l} \\
&= \sum_{m=0}^{N-1} (\overline{W_N} \otimes \overline{W_N})_{k,m} [D(W_N \otimes W_N)]_{m,l} \\
&= \sum_{m=0}^{N-1} (\overline{W_N} \otimes \overline{W_N})_{k,m} \sum_{n=0}^{N-1} D_{m-n} (W_N \otimes W_N)_{n,l} \\
&= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \overline{W_N(k,m)} W_N(n,l) \overline{W_N} D_{m-n} W_N \\
&= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{-2\pi j \frac{km-ln}{N}} \overline{W_N} D_{m-n} W_N \\
&= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n'=0}^{N-1} e^{-2\pi j \frac{km-l(m-n')}{N}} \overline{W_N} D_{n'} W_N \\
&= \frac{1}{N} \sum_{m=0}^{N-1} e^{-2\pi j \frac{m(k-l)}{N}} \sum_{n'=0}^{N-1} e^{-2\pi j \frac{ln'}{N}} \overline{W_N} D_{n'} W_N \\
&= \delta(k-l) \sum_{n=0}^{N-1} e^{-2\pi j \frac{ln}{N}} \overline{W_N} D_n W_N.
\end{aligned}$$

Hence for any $k, l \in \{0, 1, \cdots, N-1\}$ and $x, y \in \{0, 1, \cdots, N-1\}$,

$$\begin{aligned}
&[(\overline{W_N} \otimes \overline{W_N}) D (W_N \otimes W_N)](x + kN, y + lN) \\
&= [(\overline{W_N} \otimes \overline{W_N}) D (W_N \otimes W_N)]_{k,l}(x, y) \\
&= \begin{cases} \sum_{n=0}^{N-1} e^{-2\pi j \frac{ln}{N}} [\overline{W_N} D_n W_N](x, y) & \text{if } k = l, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \sum_{n=0}^{N-1} e^{-2\pi j \frac{ln}{N}} \sum_{m=0}^{N-1} D_{n,m} e^{-2\pi j \frac{mx}{N}} & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} D(m + nN, 0) e^{-2\pi j (\frac{mx}{N} + \frac{ln}{N})} & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} h(m, n) e^{-2\pi j (\frac{mx}{N} + \frac{kn}{N})} & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} N^2\, DFT(h)(x, k) & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

(b) Let $\vec{f} = \mathcal{S}(f)$ and $\vec{g} = \mathcal{S}(g)$.

$$(\lambda D^T D + L^T L)\vec{f} = \lambda D^T \vec{g}.$$

Since D and L are block-circulant, the result from (a). asserts that

$$[(\overline{W_N} \otimes \overline{W_N}) D (W_N \otimes W_N)](x + kN, y + lN) = \begin{cases} N^2 DFT(h)(x, k) & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise} \end{cases};$$

equivalently, denoting the $N^2 \times N^2$ diagonal matrix by Λ_D,

$$D = (W_N \otimes W_N) \Lambda_D (\overline{W_N} \otimes \overline{W_N}).$$

Similarly,

$$L = (W_N \otimes W_N) \Lambda_L (\overline{W_N} \otimes \overline{W_N}).$$

Hence

$$\begin{aligned}
(\lambda D^T D + L^T L)\vec{f} &= (\lambda D^* D + L^* L)\vec{f} \\
&= \{\lambda [(W_N \otimes W_N) \Lambda_D (\overline{W_N} \otimes \overline{W_N})]^* (W_N \otimes W_N) \Lambda_D (\overline{W_N} \otimes \overline{W_N}) \\
&\quad + [(W_N \otimes W_N) \Lambda_L (\overline{W_N} \otimes \overline{W_N})]^* (W_N \otimes W_N) \Lambda_L (\overline{W_N} \otimes \overline{W_N})\}\vec{f} \\
&= [\lambda (\overline{W_N} \otimes \overline{W_N})^* \Lambda_D^* \Lambda_D (\overline{W_N} \otimes \overline{W_N}) + (\overline{W_N} \otimes \overline{W_N})^* \Lambda_L^* \Lambda_L (\overline{W_N} \otimes \overline{W_N})]\vec{f} \\
&= (W_N \otimes W_N)(\lambda \Lambda_D^* \Lambda_D + \Lambda_L^* \Lambda_L)(\overline{W_N} \otimes \overline{W_N})\vec{f} \\
&= (W_N \otimes W_N)(\lambda \Lambda_D^* \Lambda_D + \Lambda_L^* \Lambda_L) N \mathcal{S}(DFT(f));
\end{aligned}$$

on the other hand,

$$\begin{aligned}
\lambda D^T \vec{g} &= \lambda D^* \vec{g} \\
&= \lambda (\overline{W_N} \otimes \overline{W_N})^* \Lambda_D^* (W_N \otimes W_N)^* \vec{g} \\
&= \lambda (W_N \otimes W_N) \Lambda_D^* (\overline{W_N} \otimes \overline{W_N}) \vec{g} \\
&= \lambda (W_N \otimes W_N) \Lambda_D^* N\, \mathcal{S}(DFT(g)).
\end{aligned}$$

Hence $(\lambda \Lambda_D^* \Lambda_D + \Lambda_L^* \Lambda_L)\mathcal{S}(DFT(f)) = \lambda \Lambda_D^* \mathcal{S}(DFT(g))$. By comparing each pair of entries,

$$(\lambda N^4 |DFT(h)(u, v)|^2 + N^4 |DFT(p)(u, v)|^2) DFT(f)(u, v) = \lambda N^2 DFT(h)(u, v) DFT(g)(u, v),$$

which yields

$$DFT(f)(u,v) = \frac{\lambda DFT(h)(u,v)DFT(g)(u,v)}{N^2(\lambda|DFT(h)(u,v)|^2 + |DFT(p)(u,v)|^2)}.$$

7. (a)

$$E_{snake,2}(\gamma) = \int_0^{2\pi} \frac{1}{2}\|\gamma'(s)\|^2\, ds + \alpha \int_0^{2\pi} \frac{1}{2}\|\gamma''(s)\|^2\, ds + \beta \int_0^{2\pi} V(\gamma(s))\, ds,$$

Given a contour $\gamma^n(t)$ at the n-th iteration. We proceed to perturb $\gamma^{n+1}(t)$ by $\gamma^{n+1} := \gamma^n + \varepsilon\varphi$ to minimize $E_{snake,2}$. We need to find φ s.t. $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{snake,2}(\gamma^{n+1}) < 0$.

$$\begin{aligned}
&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{snake,2}(\gamma^{n+1}) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_0^{2\pi} \frac{1}{2}\|(\gamma^n)'(s) + \varepsilon\varphi'(s)\|^2\ ds\ + \alpha \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_0^{2\pi} \frac{1}{2}\|(\gamma^n)''(s) + \varepsilon\varphi''(s)\|^2\ ds \\
&\quad + \beta \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_0^{2\pi} V(\gamma^n(s) + \varepsilon\varphi(s))\ ds \\
&= \int_0^{2\pi} (\gamma^n)'(s)\cdot\varphi'(s)\ ds\ + \alpha \int_0^{2\pi} (\gamma^n)''(s)\cdot\varphi''(s)\ ds\ + \beta \int_0^{2\pi} \nabla V(\gamma^n(s))\cdot\varphi(s)\ ds \\
&= -\int_0^{2\pi} (\gamma^n)''(s)\cdot\varphi(s)\ ds\ + \alpha \int_0^{2\pi} (\gamma^n)^{(4)}(s)\cdot\varphi(s)\ ds\ + \beta \int_0^{2\pi} \nabla V(\gamma^n(s))\cdot\varphi(s)\ ds \\
&= \int_0^{2\pi} [-(\gamma^n)''(s) + \alpha(\gamma^n)^{(4)}(s) + \beta\nabla V(\gamma^n(s))]\cdot\varphi(s)\ ds
\end{aligned}$$

In order that $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{snake,2}(\gamma^{n+1}) < 0$ (decreasing), we must have:

$$\varphi(s) = (\gamma^n)''(s) - \alpha(\gamma^n)^{(4)}(s) - \beta\nabla V(\gamma^n(s))$$

Thus, we must modify γ^n by:

$$\gamma^{n+1} = \gamma^n + \varepsilon((\gamma^n)''(s) - \alpha(\gamma^n)^{(4)}(s) - \beta\nabla V(\gamma^n(s))) \qquad \text{for small } \varepsilon > 0$$

or

$$\frac{\gamma^{n+1} - \gamma^n}{\varepsilon} = \underbrace{(\gamma^n)''(s) - \alpha(\gamma^n)^{(4)}(s) - \beta\nabla V(\gamma^n(s))}_{-\nabla E(\gamma^n(s))\ (definition)}$$

In the continuous setting, we aim to obtain a time-dependent contour: $\gamma_t(s) := \gamma(s;t)$ such that: $\frac{d}{dt}\gamma_t(s) = -\nabla E(\gamma_t(s))$.

(b) Let N = number of discrete points in $[0, 2\pi]$,
$\sigma = \frac{2\pi}{N}$ = step length and $s_i = i\sigma \quad (i = 1, 2, \cdots, N)$
$u_i^k = \gamma(s_i; t^k) = \gamma(i\sigma; k\tau)$ = i-th node of the contour, where τ is time step.

Define $u^k = (u_1^k, u_2^k, \cdots, u_N^k)^T \in M_{N\times 2}(\mathbb{R})$ = discrete closed curve / contour, where $u_i^k \in \mathbb{R}^2$ for all i.
The discrete derivative can be approximated by finite difference scheme:

$$\gamma_k'(i\sigma) = \frac{u_{i+1}^k - u_i^k}{\sigma} \text{ and } \gamma_k''(i\sigma) = \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{\sigma^2} \qquad i = 1, 2, \cdots, N$$

Here, we assume $\gamma_{N+1}^k = \gamma_1^k$; $\gamma_{N+2}^k = \gamma_2^k$; $\gamma_N^k = \gamma_0^k$; $\gamma_{N-1}^k = \gamma_{-1}^k$ (since the contour is closed).

Thus, the discrete snake energy can be written as:

$$E_{snake,2}(u) = \sum_{i=1}^{N} \frac{1}{2} \left\| \frac{u_{i+1} - u_i}{\sigma} \right\|^2 \sigma + \alpha \sum_{i=1}^{N} \frac{1}{2} \left\| \frac{u_{i+1} - 2u_i + u_{i-1}}{\sigma^2} \right\|^2 \sigma + \beta \sum_{i=1}^{N} V(u_i)\sigma$$

where $u = (u_1, u_2, \cdots, u_N)^T$ is a discrete closed curve ($u_i \in \mathbb{R}^2$ for all i). We can throw away σ

$$E_{snake,2}(u) = \sum_{i=1}^{N} \frac{1}{2} \left\| \frac{u_{i+1} - u_i}{\sigma} \right\|^2 + \alpha \sum_{i=1}^{N} \frac{1}{2} \left\| \frac{u_{i+1} - 2u_i + u_{i-1}}{\sigma^2} \right\|^2 + \beta \sum_{i=1}^{N} V(u_i)$$

(c) To minimize $E_{snake,2}$, we compute ∇E and find u such that $\nabla E(u) = 0$. now,

$$\begin{aligned}
\frac{\partial E}{\partial u_i} &= -\left(\frac{u_{i+1} - u_i}{\sigma^2}\right) + \left(\frac{u_i - u_{i-1}}{\sigma^2}\right) \\
&\quad + \frac{\alpha}{2}\left[2\frac{u_i + u_{i-2} - 2u_{i-1}}{\sigma^4} + 2\frac{u_{i+2} + u_i - 2u_{i+1}}{\sigma^4} - 4\frac{u_{i+1} + u_{i-1} - 2u_i}{\sigma^4}\right] \\
&\quad + \beta\nabla V(u_i) \\
&= \frac{\alpha}{\sigma^4}u_{i+2} + \frac{-\sigma^2 - 4\alpha}{\sigma^4}u_{i+1} + \frac{2\sigma^2 + 6\alpha}{\sigma^4}u_i + \frac{-\sigma^2 - 4\alpha}{\sigma^4}u_{i-1} + \frac{\alpha}{\sigma^4}u_{i-2} + \beta\nabla V(u_i)
\end{aligned}$$

(Recall that: $u_i = (u_{i_1}, u_{i_2})^T \in \mathbb{R}^2$. We define: $\frac{\partial E}{\partial u_i} = \left(\frac{\partial E}{\partial u_{i_1}}, \frac{\partial E}{\partial u_{i_2}}\right)^T$.

Thus, $\frac{\partial V}{\partial u_i} = \left(\frac{\partial V}{\partial u_{i_1}}, \frac{\partial V}{\partial u_{i_2}}\right)^T = \nabla V(u_i)$)

Define:

$$D = -\begin{pmatrix}
\frac{2\sigma^2+6\alpha}{\sigma^4} & \frac{-\sigma^2-4\alpha}{\sigma^4} & \frac{\alpha}{\sigma^4} & 0 & \cdots & 0 & \frac{\alpha}{\sigma^4} & \frac{-\sigma^2-4\alpha}{\sigma^4} \\
\frac{-\sigma^2-4\alpha}{\sigma^4} & \frac{2\sigma^2+6\alpha}{\sigma^4} & \frac{-\sigma^2-4\alpha}{\sigma^4} & \frac{\alpha}{\sigma^4} & \cdots & 0 & 0 & \frac{\alpha}{\sigma^4} \\
\frac{\alpha}{\sigma^4} & \frac{-\sigma^2-4\alpha}{\sigma^4} & \frac{2\sigma^2+6\alpha}{\sigma^4} & \frac{-\sigma^2-4\alpha}{\sigma^4} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{-\sigma^2-4\alpha}{\sigma^4} & \frac{\alpha}{\sigma^4} & 0 & 0 & \cdots & \frac{\alpha}{\sigma^4} & \frac{-\sigma^2-4\alpha}{\sigma^4} & \frac{2\sigma^2+6\alpha}{\sigma^4}
\end{pmatrix}$$

Define $F(u) = (F_1(u), F_2(u), \cdots, F_N(u))^T \in M_{N\times 2}(\mathbb{R})$
where $F_i(u) = -\nabla V(u_i), \quad i = 1, 2, \cdots, N.$

Then: $\frac{\partial E}{\partial u_i} = -(Du)_i - \beta(F(u))_i$

Using the gradient descent method, we can minimize E_{snake} by Explicit Euler scheme:

$$\frac{u_i^{k+1} - u_i^k}{\tau} = (Du^k)_i + \beta(F(u^k))_i \qquad \tau = \text{ time step}$$

8. (a)

$$\begin{aligned}
\frac{\partial E_{snake}}{\partial \mathbf{u}_{i,j}} &= \frac{1}{2}\frac{\partial}{\partial \mathbf{u}_{i,j}}\left[\sum_{p=1}^{N}\left|\frac{\mathbf{u}_{p+1} - \mathbf{u}_p}{\sigma}\right|^2 + \beta\sum_{p=1}^{N} V(\mathbf{u}_p)\right] \\
&= \frac{-\mathbf{u}_{i+1,j} + 2\mathbf{u}_{i,j} - \mathbf{u}_{i-1,j}}{2\sigma^2} + \frac{\beta}{2}\sum_{p=1}^{N}\frac{\partial V(\mathbf{u}_p)}{\partial \mathbf{u}_{i,j}}.
\end{aligned}$$

Hence the explicit Euler scheme is the steepest descent scheme for E, and thus given τ is sufficiently small, $E_{snake}(\mathbf{u}^{k+1}) \le E_{snake}(\mathbf{u}^k)$ for $k = 0, 1, 2, \ldots$.

(b) Suppose $\mathbf{u}_m^k = r_k(\cos(\theta_k + \frac{2\pi m}{N}), \sin(\theta_k + \frac{2\pi m}{N}))^T \sim r_k e^{j(\theta_k + \frac{2\pi m}{N})}$ for some $k \in \mathbb{N} \cup \{0\}$ and $r_k, \theta_k \in \mathbb{R}$. Then

$$\begin{aligned}\mathbf{u}_{m+1}^k + \mathbf{u}_{m-1}^k &\sim r_k e^{j(\theta_k + \frac{2\pi(m+1)}{N})} + r_k e^{j(\theta_k + \frac{2\pi(m-1)}{N})} \\ &= 2r_k e^{j(\theta_k + \frac{2\pi m}{N})} \cos\frac{\pi m}{N} \\ &\sim 2\cos\frac{2\pi m}{N}\mathbf{u}_m^k,\end{aligned}$$

and thus

$$\begin{aligned}\mathbf{u}_m^{k+1} &= \mathbf{u}_m^k + \frac{\tau}{\sigma^2}(\mathbf{u}_{m+1}^k - 2\mathbf{u}_m^k + \mathbf{u}_{m-1}^k) - \beta\tau(\frac{\partial V(\mathbf{u}_m^k)}{\partial \mathbf{u}_{m,1}^k}, \frac{\partial V(\mathbf{u}_m^k)}{\partial \mathbf{u}_{m,2}^k})^T \\ &= [1 + \frac{2\tau}{\sigma^2}(\cos\frac{2\pi m}{N} - 1) - 2\beta\tau]\mathbf{u}_m^k,\end{aligned}$$

and thus $\mathbf{u}^{k+1}$ is also a discrete curve on a circle centred at the origin with equally-spaced nodes.

Since $\mathbf{u}_m^0 = (\cos\frac{2\pi m}{N}, \sin\frac{2\pi m}{N})^T$ satisfies the induction hypothesis, $\mathbf{u}^k$ is a discrete curve representing a circle for $k = 0, 1, 2, \ldots$.

$$\begin{aligned}\frac{\mathbf{u}_m^{k+1} - \mathbf{u}_m^k}{\tau} &= \frac{\mathbf{u}_m^{k+1} - 2\mathbf{u}_m^{k+1} + \mathbf{u}_{m-1}^{k+1}}{\sigma^2} - \beta\nabla V(\mathbf{u}_m^k) \\ \text{and thus } \mathbf{u}_m^{k+1} &= \frac{\tau}{\sigma^2}\mathbf{u}_{m+1}^k + (1 - \frac{2\tau}{\sigma^2} - 2\beta\tau)\mathbf{u}_m^k + \frac{\tau}{\sigma^2}\mathbf{u}_{m-1}^k.\end{aligned}$$

In matrix form,

$$\begin{pmatrix}\mathbf{u}_{1,1}^{k+1} & \mathbf{u}_{1,2}^{k+1} \\ \vdots & \vdots \\ \vdots & \vdots \\ \mathbf{u}_{N,1}^{k+1} & \mathbf{u}_{N,2}^{k+1}\end{pmatrix} = \begin{pmatrix}1 - \frac{2\tau}{\sigma^2} - 2\beta\tau & \frac{\tau}{\sigma^2} & & & \frac{\tau}{\sigma^2} \\ \frac{\tau}{\sigma^2} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ \frac{\tau}{\sigma^2} & & & \ddots & \ddots\end{pmatrix}\begin{pmatrix}\mathbf{u}_{1,1}^k & \mathbf{u}_{1,2}^k \\ \vdots & \vdots \\ \vdots & \vdots \\ \mathbf{u}_{N,1}^k & \mathbf{u}_{N,2}^k\end{pmatrix}.$$

Since the matrix (hereafter denoted by A) is circulant, its eigenvalues are known to be

$$\lambda_l = 1 - \frac{2\tau}{\sigma^2} - 2\beta\tau + \frac{\tau}{\sigma^2}e^{2\pi\sqrt{-1}\frac{l}{N}} + \frac{\tau}{\sigma^2}e^{-2\pi\sqrt{-1}\frac{l}{N}} = 1 - \frac{2\tau}{\sigma^2}(1 - \cos\frac{2\pi l}{N}) - 2\beta\tau \le 1 - 2\beta\tau,$$

which has $|\lambda_l| < 1$ given $0 < \tau < \frac{1}{\beta}$.

Hence given $0 < \tau < \frac{1}{\beta}$, $\mathbf{u}^k$ converges to the origin for any initial $\mathbf{u}^0$.