Lecture 9:

Recall: Math. formulation for image blur

observed

original

$$g = \underbrace{\tilde{h} * f}_{\text{Blur}} + \underbrace{n}_{\text{noise}}$$

$$\Big\downarrow \text{DFT}$$

$$G(u,v) = \underbrace{c\,\tilde{H}}_{H}(u,v)\, F(u,v) + N(u,v)$$

$$\Big\downarrow \text{iDFT}$$

$$f$$

Image deblurring in the frequency domain: (Assume H is known)

Method 1: Direct inverse filtering

Let $T(u,v) = \dfrac{1}{H(u,v) + \underbrace{\varepsilon\, \mathrm{sgn}(H(u,v))}_{\text{Avoid singularity}}}$ ($\mathrm{sgn}(z) = 1$ if $\mathrm{Re}(z) \geq 0$ and $\mathrm{sgn}(z) = -1$ otherwise)

Compute $\hat{F}(u,v) = G(u,v)\, T(u,v)$.

Find inverse DFT of $\hat{F}(u,v)$ to get an image $\hat{f}(x,y)$.

Good: Simple

Bad: Boast up noise

$$\hat{F}(u,v) = \underbrace{G(u,v)}_{= H(u,v)F(u,v) + N(u,v)} T(u,v) \approx F(u,v) + \frac{N(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$$

Note: $H(u,v)$ is big for (u,v) close to $(0,0)$ (Keep low frequencies)
is small for (u,v) far away from $(0,0)$

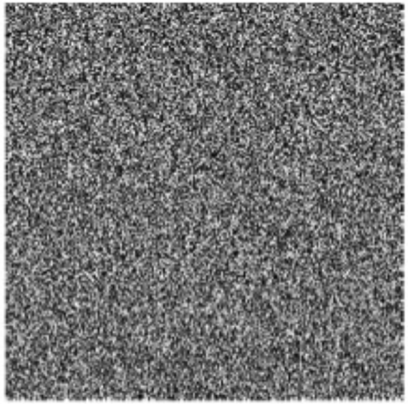
$\therefore \dfrac{N(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$ is big for (u,v) far away from $(0,0)$ — Large gain in high frequencies

$\Downarrow$

Boast up noises!!

Original

Blurred image



Direct inverse filtering

Method 2: Modified inverse filtering

Let $B(u,v) = \dfrac{1}{1 + \left(\dfrac{u^2+v^2}{D^2}\right)^n}$ and $T(u,v) = \dfrac{B(u,v)}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$.

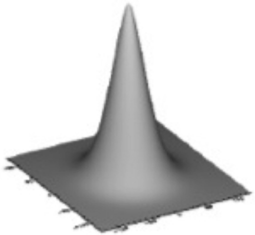
Then: define: $\hat{F}(u,v) = T(u,v)\,G(u,v) \approx F(u,v)\,B(u,v) + \dfrac{N(u,v)\,B(u,v)}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$

$\dfrac{N(u,v)\,B(u,v)}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))} \approx \dfrac{N(u,v)}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$ for $(u,v) \approx (0,0)$

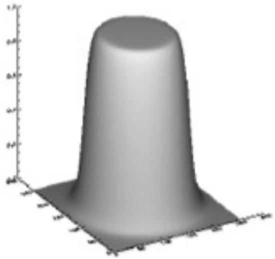
$\dfrac{N(u,v)\,B(u,v)}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$ is small (as $B(u,v)$ is small) for (u,v) far away from $(0,0)$.

$\dfrac{B(u,v)}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$ suppresses the high-frequency gain.

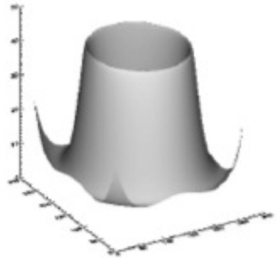
Bad: Has to choose D and n carefully.



$H(u,v)$



$B(u,v)$: $\mathsf{D} = 90, n = 8$



Inverse B/H

Original Image $G(u,v)$

Blurred using $\mathsf{D} = 90, n = 8$

Restored with a best D and n.

Method 3: Wiener filter

Let $T(u,v) = \dfrac{\overline{H(u,v)}}{|H(u,v)|^2 + \dfrac{S_n(u,v)}{S_f(u,v)}}$ where $S_n(u,v) = |N(u,v)|^2$

$S_f(u,v) = |F(u,v)|^2$

If $S_n(u,v)$ and $S_f(u,v)$ are not known, then we let $K = \dfrac{S_n(u,v)}{S_f(u,v)}$ to get:

$$T(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + K}$$

Let $\hat{F}(u,v) = T(u,v)\,G(u,v)$. Compute $\hat{f}(x,y)$ = inverse DFT of $\hat{F}(u,v)$.

In fact, the Wiener filter can be described as an inverse filtering as follows:

$$\hat{F}(u,v) = \left[\left(\frac{1}{H(u,v)}\right)\left(\frac{|H(u,v)|^2}{|H(u,v)|^2 + K}\right)\right] G(u,v)$$

Behave like "Modified inverse filtering"

≈ 0 if $H(u,v) \approx 0$ (if (u,v) far away from 0)

≈ 1 if $H(u,v)$ is large (if $(u,v) \approx (0,0)$)

What does Wiener filtering do mathematically?

We can show: Wiener filter minimizes the mean square error:

$$\varepsilon^2(\underset{\substack{\uparrow \\ \text{original}}}{f}, \underset{\substack{\uparrow \\ \text{Restored}}}{\hat{f}}) = \iint |f(x,y) - \hat{f}(x,y)|^2 \, dx \, dy$$

(Sketch of proof)

(We assume the continuous case to avoid complicated indices)

$$\text{Let } \overset{\text{Observed}}{g} = \overset{\text{degradation}}{h} * \underset{\text{original}}{f} + \overset{\text{noise}}{n}$$

Let $\hat{f}$ be the restored image.

Define: $\hat{f}(x,y) = w(x,y) * g(x,y)$ for some $w(x,y)$

(FT of $\hat{f}$ is like: $W(u,v)\,G(u,v)$)

Goal: Find $W(u,v)$ such that $\varepsilon^2(f, \hat{f})$ is minimized.

Recall: $\hat{f}$ is obtained as follows:

Step 1: Let $\hat{F}(u,v) = \underbrace{W(u,v)}_{\text{Filter}} G(u,v)$

Step 2: Compute iFT of $\hat{F}$ to get $\hat{f}$

$\therefore \hat{f} = w * g$ for some w.

(Sketch of proof)

We need to use: Parseval Theorem:

$$\mathcal{E}^2(f,\hat{f}) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y) - \hat{f}(x,y)|^2 \, dx\, dy = C \iint |F(u,v) - \hat{F}(u,v)|^2 \, du\, dv$$ for some constant C

where $F(u,v) = DFT(f)$, $\hat{F}(u,v) = DFT(\hat{f})$

So, $\hat{F}(u,v) = W(u,v)\, G(u,v) = W(u,v)\big(H(u,v)\, F(u,v) + N(u,v)\big)$

In other words, $F - \hat{F} = (1 - WH)F - WN$

and $\mathcal{E}^2(f,\hat{f}) = C \iint |(1-WH)F - WN|^2 \, du\, dv$

$$= C \iint |(1-WH)F|^2 + |WN|^2$$ (if f and n are spatially-uncorrelated)

$\mathcal{E}^2$ is dependent on W.

To minimize $\mathcal{E}^2(W)$, we consider:

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{E}^2(W + tV) = 0 \text{ for all } V.$$

We get: $\iint -(1-\overline{W}\overline{H})H|F|^2 V - (1-WH)\overline{H}|F|^2\overline{V} + \overline{W}|N|^2 V + W|N|^2\overline{V} = 0$ for all V.

Put $V = -(1-WH)\overline{H}|F|^2 + W|N|^2$. Then: we have: $\iint \left| -(1-WH)\overline{H}|F|^2 + W|N|^2 \right|^2 du\, dv = 0.$

$$\therefore \quad -(1-WH)\overline{H}|F|^2 + W|N|^2 = 0$$

$$\Downarrow$$

$$W = \frac{\overline{H}}{|H|^2 + |N|^2/|F|^2}.$$

Method 4: Constrained least square filtering

Disadvantages of Wiener's filter:

① $|N(u,v)|^2$ and $|F(u,v)|^2$ must be known / guessed

② Constant estimation of ratio is not always suitable

Goal: Consider a least square minimization model.

Let $g = \underset{\text{degradation}}{h} * f + \underset{\text{noise}}{n}$

In matrix form, $\underset{\mathcal{S}(g)}{\underset{\shortparallel}{\vec{g}}} = D\,\underset{\mathcal{S}(f)}{\underset{\shortparallel}{\vec{f}}} + \underset{\mathcal{S}(n)}{\underset{\shortparallel}{\vec{n}}}$ $\quad \vec{g}, \vec{f}, \vec{n} \in \mathbb{R}^{N^2}$, $D \in M_{N^2 \times N^2}$

D: transformation matrix of $h * f$ (or f)

Given $\vec{g}$, we need to find an estimation of $\vec{f}$ such that it minimizes:

$$E(f) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} |\nabla^2 f(x,y)|^2 \text{ subject to the constraint: }$$

$$\|\vec{g} - D\vec{f}\|^2 = \varepsilon$$

- $\displaystyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} |\nabla^2 f(x,y)|^2 \longleftarrow$ Denoise
- $\|\vec{g} - D\vec{f}\|^2 = \varepsilon \longleftarrow$ Deblur

In the discrete case, we can estimate:

$$\nabla^2 f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)$$

Taylor expansion:

$$\frac{\partial^2 f(x,y)}{\partial x^2} \approx \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) \approx \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2}$$

$$\xrightarrow{\text{Put } h=1} \quad \nabla^2 f(x,y) = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)(x,y)$$

More generally, $\nabla^2 f = p * f \longleftarrow$ discrete convolution

where

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ & 1 & -4 & 1 & \\ \vdots & & 1 & & \vdots \\ 0 & & \cdots & & 0 \end{pmatrix}$$

<u>Remark:</u> $\underbrace{\|\vec{g} - D\vec{f}\|^2}_{= \|\vec{n}\|^2} = \varepsilon$ means we allow some fixed level of noise.

Our problem:

minimize = $\|L\vec{f}\|^2$ subject to $\|\vec{g} - D\vec{f}\|^2 = \varepsilon$

$$\|L\vec{f}\|^2 = (L\vec{f})^T(L\vec{f}) = \vec{f}^{\,T} L^T L \vec{f}$$

$$\vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N^2} \end{pmatrix}$$

Solve:

Constrained N^2 variable optimization problem

(Advanced calculus!!)