Discrete Fourier Transform:

Definition: The 1D discrete Fourier Transform (DFT) of a function $f(k)$, defined at discrete points $k = 0, 1, 2, \ldots, N-1$ is defined as:

$$\hat{f}(m) = \frac{1}{N} \sum_{k=0}^{N-1} f(k)\, e^{-j\frac{2\pi m k}{N}}$$

(where $j = \sqrt{-1}$, $e^{j\theta} = \cos\theta + j\sin\theta$)

The 2D DFT of a $M \times N$ image $g = (g(k,l))_{k,l}$, where $0 \le k \le M-1$, $0 \le l \le N-1$ is defined as:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

Remark: The inverse of DFT is given by:

$$g(p,q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{g}(m,n)\, e^{j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)}$$

DFT of g

(no $\frac{1}{Mn}$!)

(no -ve sign)

DFT of convolution:

Recall: $g * w(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m')\, w(n', m')$ $\quad (g, m \in M_{N\times M}(\mathbb{R}))$

Then, the DFT $(g * w)(p,q) = MN\, \text{DFT}(g)(p,q)\, \text{DFT}(w)(p,q)$ for all $0 \le p \le N-1$, $0 \le q \le M-1$

$\therefore$ DFT of convolution can be reduced to simple multiplication!

Recall: Shift-invariant image transformation = 2D convolution.

$\therefore$ Easy computation/manipulation of shift-invariant transf. after DFT!!

Understanding convolution:

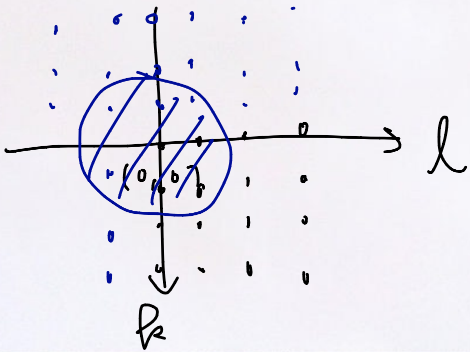
Recall: Discrete convolution:

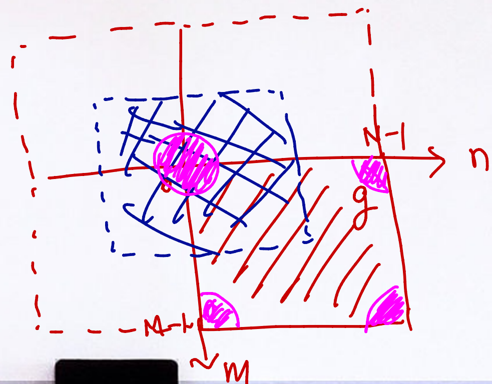
$$v(n,m) = \underbrace{\sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m')\, I(n', m')}_{\text{Linear combination of pixel values of } I}$$

$$\shortparallel$$

$$g * I(n,m)$$

In particular, if $g(k,l)$ is only non-zero around $(0,0)$, then, $g * I(n,m)$ is a linear combination of pixel value of I around (n,m)!!





Example: Suppose g looks like the following:

$$g = \begin{array}{cl} \begin{pmatrix} & 1 & 2 & 3 & \\ 0 & 4 & 5 & 6 & 0 \\ & 7 & 8 & 9 & \end{pmatrix} & \begin{array}{l} \leftarrow k=-1 \\ \leftarrow k=0 \\ \leftarrow k=1 \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ l=-1 & l=0 & l=1 \end{array} & \end{array}$$

$$I * g(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m')\, I(n', m')$$

$$\text{Linear combination of neighborhood pixel values} \left\{ \begin{aligned} &= 1 \cdot I(n+1, m+1) + 2 \cdot I(n+1, m) + 3 \cdot I(n+1, m-1) \\ &+ 4 \cdot I(n, m+1) + 5 \cdot I(n, m) + 6 \cdot I(n, m-1) \\ &+ 7 \cdot I(n-1, m+1) + 8 \cdot I(n-1, m) + 9 \cdot I(n-1, m-1) \end{aligned} \right.$$

Note:

(Spatial domain) $\quad I * g \quad$ (Linear filtering: Linear combination of neighborhood pixel values)

$$\Big\downarrow \text{DFT}$$

(Frequency domain) $\quad MN\,\underbrace{\hat{I} \odot \hat{g}}_{\text{pixel-wise multiplication}} \quad$ (Modifying the Fourier coefficients by multiplication)

Image enhancement in the frequency domain:

Goal: 1. Remove $\underbrace{\text{high-frequency components}}_{\text{noise}}$ (low-pass filter) for image denoising.

2. Remove $\underbrace{\text{low-frequency components}}_{\text{non-edge}}$ (high-pass filter) for the extraction of image details.

High/Low frequency components of $\hat{F}$

Let F be a $N \times N$ image, N = even. Let $\hat{F}$ = DFT of F.

$$\therefore \quad \hat{F}(k, l) = \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} F(m,n)\, e^{-j 2\pi \cdot \left(\frac{mk + nl}{N}\right)}$$

Fourier coefficients of F at (k, l)

Observe that : for $0 \leq k, l \leq \frac{N}{2} - 1$

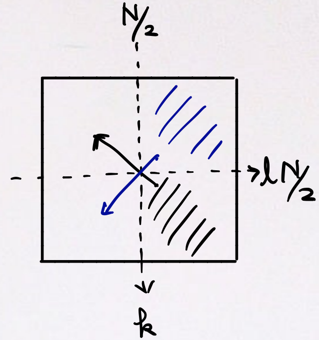
$$\hat{F}\left(\frac{N}{2} + k, \frac{N}{2} + l\right) = \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} F(m,n)\, e^{-j \frac{2\pi}{N}\left(m\left(\frac{N}{2} + k\right) + n\left(\frac{N}{2} + l\right)\right)}$$

$$= \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} F(m,n)\, (-1)^{m+n}\, e^{-j \frac{2\pi}{N}(m(-k) + n(-l))}$$

$$= \frac{1}{N^2} \overline{\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} F(m,n)\, e^{-j\frac{2\pi}{N}\left(m\left(\frac{N}{2}-k\right)+n\left(\frac{N}{2}-l\right)\right)}}$$

$$= \overline{\hat{F}\left(\frac{N}{2}-k,\ \frac{N}{2}-l\right)}$$

$\therefore$ Computing part of $\hat{F}$ can determine the rest !!



Observation:

1. When k and l are close to $N/2$, $\hat{F}\left(\frac{N}{2}+k, \frac{N}{2}+l\right)$ is associated to $e^{+j\frac{2\pi}{N}\left(\left(\frac{N}{2}+k\right)m+\left(\frac{N}{2}+l\right)n\right)}$

$$\frac{N}{2}+k \approx N, \quad \frac{N}{2}+l \approx N$$

$$\approx e^{j\frac{2\pi}{N}(k'm+l'n)} \quad \text{where } (k',l') \approx (0,0)$$

$$= \cos\left(\frac{2\pi}{N}(k'm+l'n)\right) + i\sin\left(\frac{2\pi}{N}(k'm+l'n)\right)$$

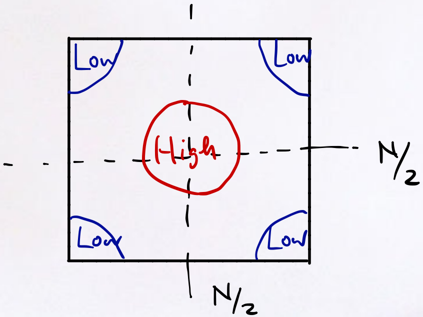
Low-frequency if $(k,l) \approx (0,0)$

$\therefore$ Fourier coefficients at the bottom right are associated to low frequency components!

2. Similarly, we can check that Fourier coefficients at the 4 corners are associated to low frequency components.

3. Fourier coefficients in the middle are associated to high-frequency components

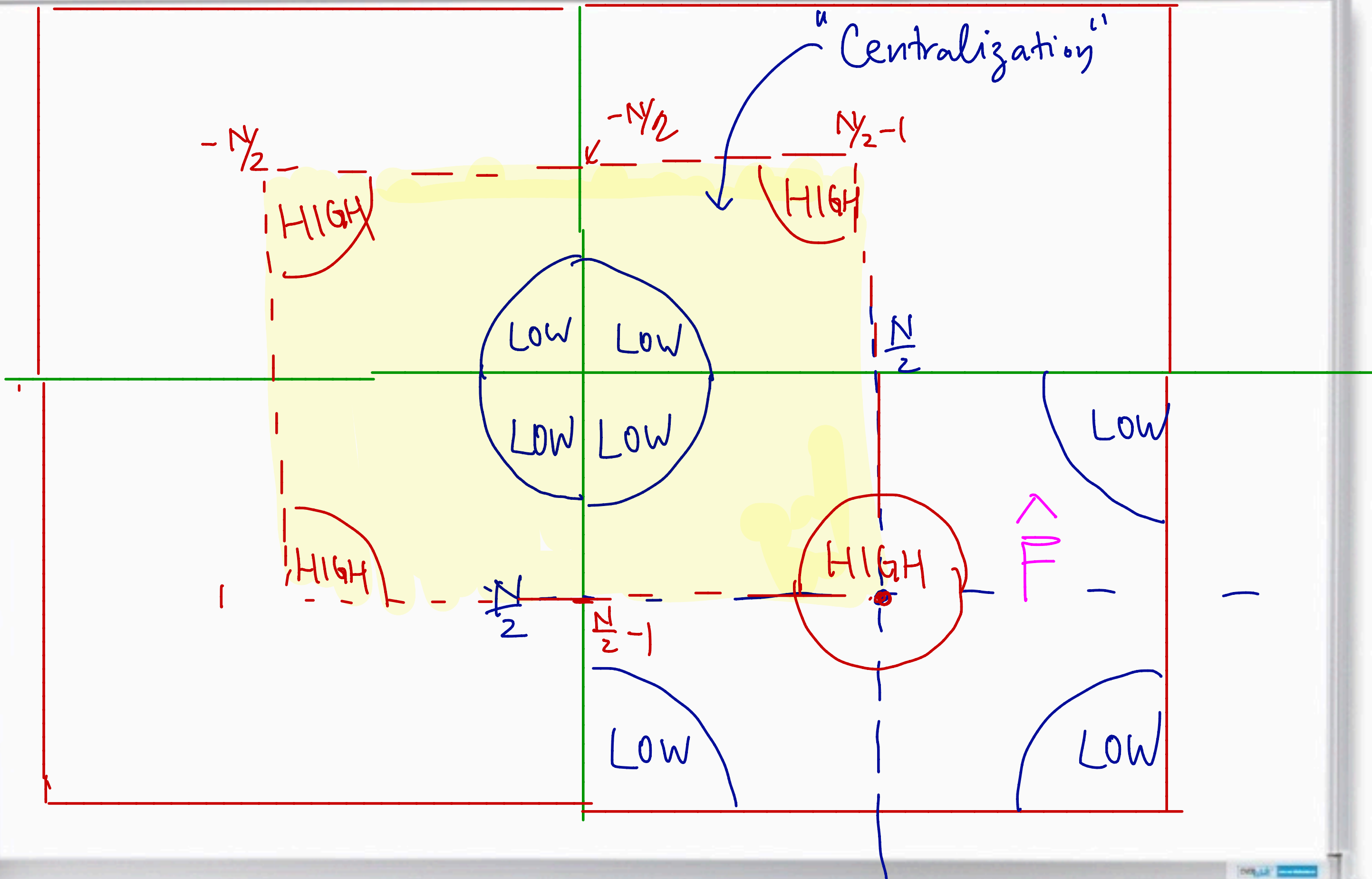
$$e^{j\frac{2\pi}{N}\left(\frac{N}{2}m+\frac{N}{2}n\right)} = e^{j\pi(m+n)} = (-1)^{m+n}$$



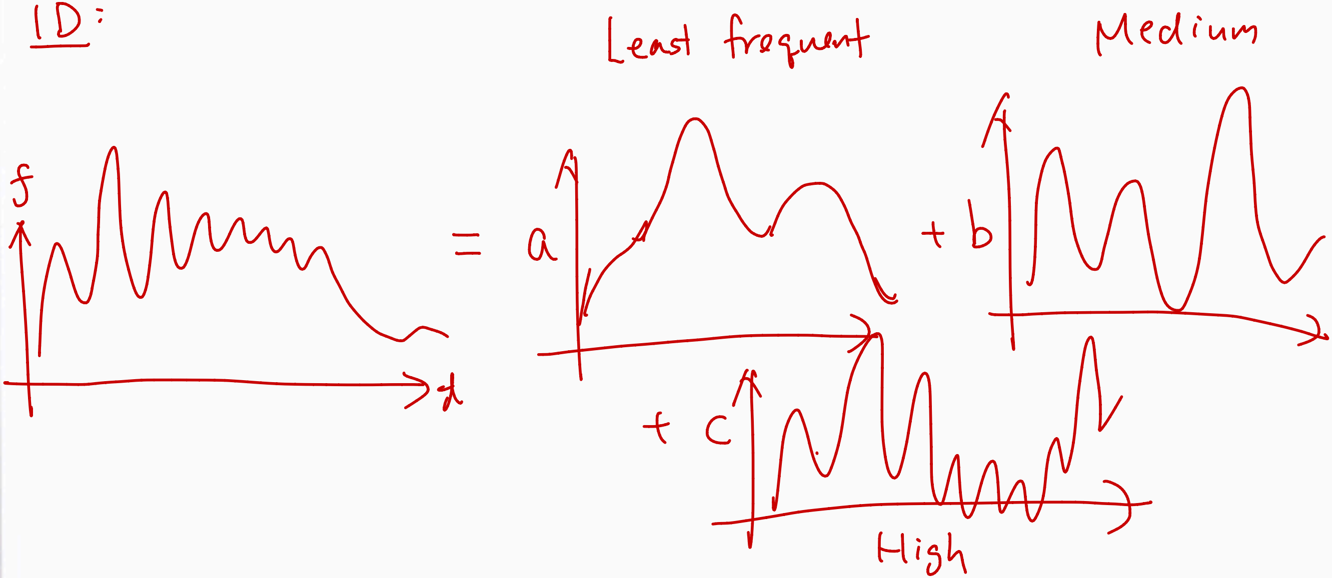
!!

$\therefore$ High-pass filtering = Remove coefficients at 4 corners

Low-pass filtering = Remove coefficients at the center


"Centralization"
-N/2
-N/2
N/2 -1
HIGH
HIGH
HIGH
HIGH
LOW
LOW
LOW
LOW
LOW
LOW
LOW
N/2
N/2
N/2 -1
F̂

1D:



To remove noise, truncate c (let c = 0)

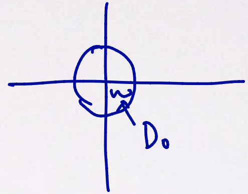
Example of Low-pass filters for image denoising

Assume that we work on the centered spectrum!

That is, consider $\hat{F}(u,v)$ where $-N/2 \le u \le N/2 - 1$, $-N/2 \le v \le N/2 - 1$.

1 Ideal low pass filter (ILPF):

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) := u^2+v^2 \le D_0^2 \\ 0 & \text{if } D(u,v) > D_0^2 \end{cases}$$

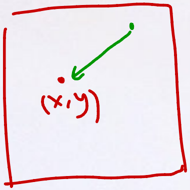


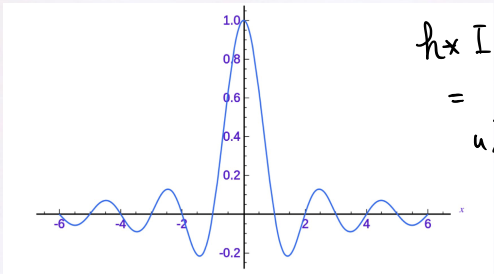
In 1-dim cross-section, $\overset{h}{\overset{\|}{\boxed{\mathcal{F}^{-1}(H(u,v))}}}$ looks like:

$$I \to \hat{I} \to \hat{I} \odot H$$

$$\downarrow \text{iDFT}$$

$$\tilde{I}$$





$$h * I(x,y) = \sum_{u,v} h(x-u, y-v)\, I(u,v)$$

every pixel values of I has an effect on $h * I(x,y)$!!

Good: Simple

Bad : Produce ringing effect!