Lecture 12:

Image denoising by solving energy minimisation problem

Consider the harmonic-L2 minimization model:

$$\text{minimize} \quad E(f) = \int_{\underbrace{\Omega}_{\text{image domain}}} \left(f(x,y) - \underbrace{g(x,y)}_{\text{Observed}}\right)^2 dx\,dy + \int \underbrace{|\nabla f|^2}_{\text{smoothness of } f} dx\,dy$$

(Look for (continuous) image f)

We find f that minimizes $E(f)$.

Take any function $v(x,y)$. Consider a real-valued function $S: \mathbb{R} \to \mathbb{R}$:

$$S(\varepsilon) := E(f + \varepsilon v) = \int_\Omega \left(f(x,y) + \varepsilon v(x,y) - g(x,y)\right)^2 dx\,dy + \int_\Omega |\nabla f + \varepsilon \nabla v|^2 dx\,dy$$

$$\frac{d}{d\varepsilon} S(\varepsilon) = 2\int_\Omega \left(f(x,y) + \varepsilon v(x,y) - g(x,y)\right)^2 dx\,dy + 2\int_\Omega \left[\left(\frac{\partial f}{\partial x} + \varepsilon \frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x} + \left(\frac{\partial f}{\partial y} + \varepsilon \frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial y}\right] dx\,dy$$

If f is a minimizer, $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\varepsilon) = 0$ for all v ($\because S(0)$ is the minimum)

$$\therefore\ S'(0) = 0 = 2\int_\Omega \big(f(x,y) - g(x,y)\big)\, v(x,y)\, dx\,dy + 2\int_\Omega \left(f_x v_x + f_y v_y\right) dx\,dy \quad \text{for all } v.$$

Remark: If we can formulate the above equation as follows:

$$\int_\Omega T(x,y)\, v(x,y)\, dx\,dy = 0 \quad \text{for all } v(x,y)$$

then, we can conclude $T(x,y) = 0$ in Ω.

In our case, first term is okay!
Second term NOT okay!

$$\nabla \cdot \big(V_1(x,y), V_2(x,y)\big) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y}$$

Useful Tool: (Integration by part)

$$\int_\Omega K(x,y)\nabla f \cdot \nabla g\, dx\,dy = -\int_\Omega \Big(\overbrace{\nabla\cdot}^{\text{divergence}}(K\nabla f)\Big)\, g\, dx\,dy + \int_{\partial\Omega} g\,(K\nabla f \cdot \vec{n})\, ds$$

where $\vec{n} = (n_1, n_2)$ = outward normal on the boundary.

In our case, we get:

$$0 = \int_\Omega (f-g)\, v\, dx\,dy - \int_\Omega (\nabla\cdot\nabla f)\, v\, dx\,dy + \int_{\partial\Omega} (\nabla f \cdot \vec{n})\, v\, ds$$

all, we get : $\int_\Omega (f - g - \Delta f)\, v \, dx\, dy - \int_{\partial\Omega} (\nabla f \cdot \vec{n})\, v \, ds = 0$ for all v

We conclude : $\begin{cases} f - g - \Delta f = 0 & \text{in } \Omega \\ \nabla f \cdot \vec{n} = 0 & \text{in } \partial\Omega \end{cases}$ (PDE)

Total-variation Denoising model

Goal: Given a noisy image $g(x,y)$, we look for $f(x,y)$ that minimizes:

$$J(f) = \frac{1}{2}\int_{\Omega} \left(f(x,y) - g(x,y)\right)^2 + \underline{\lambda} \overbrace{\int_{\Omega} |\nabla f(x,y)|\, dx\, dy}^{TV}$$

Same idea: Let $s(\varepsilon) \overset{\text{def}}{=} J(f+\varepsilon v)$ (Assume f = minimizer of J)

$$= \frac{1}{2}\int_\Omega (f+\varepsilon v-g)^2 + \lambda\int_\Omega \underbrace{|\nabla f+\varepsilon\nabla v|}_{\sqrt{(\nabla f+\varepsilon\nabla v)\cdot(\nabla f+\varepsilon\nabla v)}}$$

$$\frac{d}{d\varepsilon}s(\varepsilon) = \left[\int_\Omega (f+\varepsilon v-g)\,v + \lambda\int_\Omega \frac{\nabla f\cdot\nabla v + 2\varepsilon\,\nabla v\cdot\nabla v}{\sqrt{(\nabla f+\varepsilon\nabla v)\cdot(\nabla f+\varepsilon\nabla v)}}\right]$$

$$\because\ \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} s(\varepsilon) = 0$$

$$\therefore\ 0 = \int_\Omega (f-g)\,v + \lambda\int_\Omega \frac{\nabla f\cdot\nabla v}{|\nabla f|}$$

$$0 = \int_\Omega (f-g)\,v - \lambda\int_\Omega \nabla\cdot\left(\frac{\nabla f}{|\nabla f|}\right)v + \lambda\int_{\partial\Omega}\left(\frac{\nabla f}{|\nabla f|}\cdot\vec{n}\right)v$$

We conclude: $$(f-g) - \lambda\,\nabla\cdot\left(\frac{\nabla f}{|\nabla f|}\right) = 0$$

In the discrete case,

$$J(f) = \frac{1}{2}\sum_{x=1}^{N}\sum_{y=1}^{N}(f(x,y)-g(x,y))^2+\lambda\sum_{x=1}^{N}\sum_{y=1}^{N}\sqrt{(f(x+1,y)-f(x,y))^2+(f(x,y+1)-f(x,y))^2}$$

J can be regarded as a multi-variable function depending on :
$f(1,1), f(1,2), \ldots, f(1,N), f(2,1), \ldots, f(2,N), \ldots, f(N,N)$.

If f is a minimizer, then $\frac{\partial J}{\partial f(x,y)} = 0$ for all (x,y).

$$\begin{aligned}\frac{\partial J}{\partial f(x,y)} &= (f(x,y)-g(x,y)) + \lambda\frac{2(f(x+1,y)-f(x,y))(-1)+2(f(x,y+1)-f(x,y))(-1)}{2\sqrt{(f(x+1,y)-f(x,y))^2+(f(x,y+1)-f(x,y))^2}}\\ &+\lambda\frac{2(f(x,y)-f(x-1,y))}{2\sqrt{(f(x,y)-f(x-1,y))^2+(f(x-1,y+1)-f(x-1,y))^2}}\\ &+\lambda\frac{2(f(x,y)-f(x,y-1))}{2\sqrt{(f(x+1,y-1)-f(x,y-1))^2+(f(x,y)-f(x,y-1))^2}} = 0\end{aligned}$$

By simplification:

$$f(x,y)-g(x,y)=\lambda\left\{\frac{f(x+1,y)-f(x,y)}{\sqrt{(f(x+1,y)-f(x,y))^2+(f(x,y+1)-f(x,y))^2}}\right.$$

$$\left.-\frac{f(x,y)-f(x-1,y)}{\sqrt{(f(x,y)-f(x-1,y))^2+(f(x-1,y+1)-f(x-1,y))^2}}\right\}$$

$$+\lambda\left\{\frac{f(x,y+1)-f(x,y)}{\sqrt{(f(x+1,y)-f(x,y))^2+(f(x,y+1)-f(x,y))^2}}\right.$$

$$\left.-\frac{f(x,y)-f(x,y-1)}{\sqrt{(f(x+1,y-1)-f(x,y-1))^2+(f(x,y)-f(x,y-1))^2}}\right\}$$

$$= \frac{\left.\frac{\partial f}{\partial x}\right|_{(x,y)}}{|\nabla f|_{(x,y)}}$$

$$= \frac{\left.\frac{\partial f}{\partial x}\right|_{(x-1,y)}}{|\nabla f|_{(x-1,y)}}$$

$$= \frac{\left.\frac{\partial f}{\partial y}\right|_{(x,y)}}{|\nabla f|_{(x,y)}}$$

$$= \frac{\left.\frac{\partial f}{\partial y}\right|_{(x,y-1)}}{|\nabla f|_{(x,y-1)}}$$

Discretization of $f - g = \lambda\, \nabla \cdot \left(\frac{\nabla f}{|\nabla f|}\right)$

How to minimise $J(f)$

We consider the problem of finding f that minimizes $J(f)$.

Consider a time-dependent image $f(x,y;\underbrace{t}_{\text{time}})$. Assume $f(x,y;t)$ satisfies:

$$\frac{df}{dt}(x,y,t) = -\nabla J(f(x,y,t))$$

We can show that $J(f(x,y;t))$ decreases as t increases

Note that:

$$\begin{aligned}\frac{d}{dt} J(f(x,y;t)) &= \nabla J(f(x,y;t)) \cdot \frac{d}{dt} f(x,y,t) \\ &= \nabla J(f(x,y,t)) \cdot (-\nabla J(f(x,y,t))) \\ &= -|\nabla J(f(x,y,t))|^2 \le 0\end{aligned}$$

In the discrete case,

$$\frac{f^{n+1} - f^n}{\Delta t} = -\nabla J(f^n)$$

(Gradient descent algorithm)

For the ROF model:

$$\begin{aligned}
&\frac{f^{n+1}(x,y) - f^n(x,y)}{\Delta t} \\
&= -(f^n(x,y) - g(x,y)) + \lambda \frac{f^n(x+1,y) - f^n(x,y)}{\sqrt{(f^n(x+1,y) - f^n(x,y))^2 + (f^n(x,y+1) - f^n(x,y))^2}} \\
&- \lambda \frac{f^n(x,y) - f^n(x-1,y)}{\sqrt{(f^n(x,y) - f^n(x-1,y))^2 + (f^n(x-1,y+1) - f^n(x-1,y))^2}} \\
&+ \lambda \frac{f^n(x,y+1) - f^n(x,y)}{\sqrt{(f^n(x+1,y) - f^n(x,y))^2 + (f^n(x,y+1) - f^n(x,y))^2}} \\
&- \lambda \frac{f^n(x,y) - f^n(x,y-1)}{\sqrt{(f^n(x+1,y-1) - f^n(x,y-1))^2 + (f^n(x,y) - f^n(x,y-1))^2}}
\end{aligned}$$

Discretization of ∇J

(Gradient descent algorithm for ROF)

Image segmentation

Basic idea of Image Segmentation:

Task: extract sets of points describing the boundaries/edges of objects;

Information from the image: ($I : \Omega \to \mathbb{R}$; Ω = image domain)

Edge detector: $V : \Omega \to \mathbb{R}$ such that $V(\vec{x})$ is small if $\vec{x}$ is on the edges of the object.

<u>Example 1:</u> $V(\vec{x}) = -|\nabla I(\vec{x})|$

In the discrete case, $\frac{\partial I}{\partial x}$ and $\frac{\partial I}{\partial y}$ (hence ∇I) can be computed by linear filtering with filter:

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Active contour model (Kass, Witkin, Terzopolous)

Let $I : \Omega \to \mathbb{R}$ be the image.

Goal: Find $\gamma : [0, 2\pi] \to \Omega \subseteq \mathbb{R}^2$ that encloses the boundary of the object. Also, assume $\gamma(0) = \gamma(2\pi)$.

Let $V : \Omega \to \mathbb{R}$ be the edge detector. We consider the snake model to find γ that minimizes the snake energy:

$$E_{snake}(\gamma) = \underbrace{\int_0^{2\pi} \frac{1}{2} |\gamma'(s)|^2 \, ds}_{\text{enhance smoothness of } \gamma} + \beta \underbrace{\int_0^{2\pi} V(\gamma(s)) \, ds}_{\text{Find } \gamma(s) \text{ that lies on the boundary.}}$$

Goal: Use gradient descent algorithm.

Let: $\gamma(s) = (\phi_1(s), \phi_2(s))$

$\gamma'(s) = (\phi_1'(s), \phi_2'(s))$

$$\Rightarrow |\gamma'(s)|^2 = (\phi_1'(s))^2 + (\phi_2'(s))^2$$

Start from γ^0 = initial curve (e.g. circle)

Iteratively look for $\gamma^1, \gamma^2, \dots, \gamma^n, \gamma^{n+1}, \dots$ such that:

$$E_{snake}(\gamma^{n+1}) < E_{snake}(\gamma^n)$$

Just like before: we need to:

$$\frac{\gamma^{n+1} - \gamma^n}{\Delta t} = - \nabla E_{snake}(\gamma^n)$$

??

Given γ^n, define $\gamma^{n+1} = \gamma^n + \varepsilon\varphi$ (Perturbation of γ^n)

(with $\varphi(0) = \varphi(2\pi)$)

Need to find φ such that $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{snake}(\gamma^n + \varepsilon\varphi) < 0$

(Then: $E_{snake}(\gamma^{n+1}) = E_{snake}(\gamma^n) + \varepsilon\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{snake}(\gamma^n + \varepsilon\varphi)\right) + O(\varepsilon^2)$)

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{snake}(\gamma^n + \varepsilon\varphi) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_0^{2\pi} \frac{1}{2}\big|\underbrace{\gamma^{n\prime}(s) + \varepsilon\varphi'(s)}_{(\gamma^{n\prime}+\varepsilon\varphi')\cdot(\gamma^{n\prime}+\varepsilon\varphi')}\big|^2 ds + \beta \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_0^{2\pi} V(\gamma^n_{(s)} + \varepsilon\varphi_{(s)})\, ds$$

$$= \int_0^{2\pi} (\gamma^n)'(s)\cdot\varphi'(s)\,ds + \beta\int_0^{2\pi} \nabla V(\gamma^n(s))\cdot\varphi(s)\,ds$$

$$= -\int_0^{2\pi} (\gamma^n)''(s)\cdot\varphi(s)\,ds + \underbrace{(\gamma^n)'(s)\,\varphi(s)\Big|_0^{2\pi}}_{0} + \beta\int_0^{2\pi} \nabla V(\gamma^n(s))\cdot\varphi(s)\,ds$$

$$= \int_0^{2\pi} \left(-(\gamma^n)''(s) + \beta\nabla V(\gamma^n(s))\right)\cdot\varphi(s)\,ds$$

In order that $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{snake}(\underset{\gamma^n+\varepsilon\varphi}{\underset{\shortparallel}{\gamma^{n+1}}}) < 0$ (decreasing), we choose:

$$\varphi(s) = \left((\gamma^n)''(s) - \beta \nabla V(\gamma^n(s)) \right)$$

$\therefore$ We modify γ^n by:

$$\frac{\gamma^{n+1} - \gamma^n}{\Delta t} = \underbrace{\left((\gamma^n)''(s) - \beta \nabla V(\gamma^n(s)) \right)}_{\text{denote: } -\nabla E_{snake}(\gamma^n(s))}$$

In the continuous case:

$$\frac{d}{dt} \underset{\gamma(s;t)}{\underset{\shortparallel\, \text{def}}{\gamma_t(s)}} = -\nabla E_{snake}(\gamma_t(s))$$