Lecture 10:

Recall:

Mathematical formulation for Image blur:

$$g = h * f + n \quad \text{(Spatial)}$$

Observed blurry — g

Degradation — h

original — f

noise — n

$$G = H \odot F + N \quad \text{(Frequency)}$$

$G = DFT(g)$; $H = DFT(h)$; $F = DFT(f)$; $N = DFT(n)$

3 methods

1. Direct Inverse filtering : $T(u,v) = \dfrac{1}{H(u,v) + \varepsilon \, \mathrm{sgn}(H(u,v))}$

   (Boast up noise)

2. Modified inverse filtering : $T(u,v) = \dfrac{B(u,v)}{H(u,v) + \varepsilon \, \mathrm{sgn}(H(u,v))}$

   Butterworth

3. Wiener filter : $T(u,v) = \dfrac{\overline{H(u,v)}}{|H(u,v)|^2 + K}$

   $\overline{H(u,v)} = \dfrac{H\bar{H}}{H}$

   $K = \dfrac{|N(u,v)|^2}{|F(u,v)|^2}$

   approximate Noise to Signal ratio

$$T(u,v) = \left(\frac{1}{H(u,v)}\right)\left(\frac{|H(u,v)|^2}{|H(u,v)|^2 + K}\right)$$

(another modified inverse filtering)

Method 4: Constrained least square filtering

Disadvantages of Wiener's filter:

① $|N(u,v)|^2$ and $|F(u,v)|^2$ must be known / guessed

② Constant estimation of ratio is not always suitable

Goal: Consider a least square minimization model.

Let $g = \underset{\text{degradation}}{h} * f + \underset{\text{noise}}{n}$

In matrix form, $\vec{g} = D\vec{f} + \vec{n}$ $\Rrightarrow$ $\vec{n} = \vec{g} - D\vec{f}$ $\Rrightarrow$ $\|\vec{n}\|^2 = \|\vec{g} - D\vec{f}\|^2$

$\vec{g} = \mathcal{S}(g)$, $\vec{f} = \mathcal{S}(f)$, $\vec{n} = \mathcal{S}(n)$

D: transformation matrix of $h*f$

Stack/vectorize an image g $\longrightarrow \mathcal{S}(g)$

$\vec{g}, \vec{f}, \vec{n} \in \mathbb{R}^{N^2}$, $D \in M_{N^2 \times N^2}$

Given $\vec{g}$, we need to find an estimation of $\vec{f}$ (or f) such that it minimizes:

$$E(f) = \underbrace{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} |\nabla^2 f(x,y)|^2}_{\text{Denoising}}$$

subject to the constraint:

$$\underbrace{\|\vec{g} - D\vec{f}\|^2 = \varepsilon}_{\text{Deblurring}}$$

- $\displaystyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} |\nabla^2 f(x,y)|^2 \longleftarrow$ Denoise
- $\|\vec{g} - D\vec{f}\|^2 = \varepsilon \longleftarrow$ Deblur

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

In the discrete case, we can estimate:

$$\nabla^2 f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y) \quad \text{for } 0 \le x,y \le N-1$$

$$= p(-1,0)f(x-(-1),y-0) + \dots + p(0,0)f(x-0,y-0)$$

Taylor expansion:

$$\frac{\partial^2 f(x,y)}{\partial x^2} \approx \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) \approx \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2}$$

$\xrightarrow{\text{Put } h=1}$ $\nabla^2 f(x,y)\ \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)(x,y)$

$$p * f(x,y) = \sum p(m,n) f(x-m, y-n)$$

More generally, $\nabla^2 f = p * f \longleftarrow$ discrete convolution

where

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ & 1 & -4 & 1 & \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix} \quad 0$$

0

Remark: $\|\vec{g} - D\vec{f}\|^2 = \varepsilon$ means we allow some fixed level of noise.

$\overset{\shortparallel}{\|\vec{n}\|^2}$

$$f(x+h) = f(x) + f'(x)\,h + \frac{f''(x)\,h^2}{2!} + \dots$$

$$+\ f(x-h) = f(x) - f'(x)\,h + \frac{f''(x)\,h^2}{2!} + \dots$$

$$f(x+h) + f(x-h) = 2f(x) + f''(x)\,h^2$$

$$\Rightarrow \quad f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Assume $S(p * f) = L\vec{f}$ ← transformation matrix representing the convolution with p.

Then: $E(\vec{f}) = (L\vec{f})^T(L\vec{f})$

We will prove:

<u>Theorem:</u> The constrained least square problem has the optimal solution in the spatial domain that satisfies:

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g}$$

for some suitable parameter γ.

In the frequency domain,

$$\hat{F}(u,v) := DFT(f)(u,v) = \frac{1}{N^2}\frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma |P(u,v)|^2} G(u,v)$$

$$\left(H = DFT(h);\ G(u,v) = DFT(g);\ P(u,v) = DFT(p) \text{ where } p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix} \right)$$

Remark: Constrained least square filtering:

observed image

$$T(u,v) = \frac{1}{N^2} \frac{H(u,v)}{|H(u,v)|^2 + \gamma |P(u,v)|^2}$$

DFT(g)

Let $\tilde{F}(u,v) = T(u,v)\, G(u,v)$

Compute Inverse DFT of $\tilde{F}(u,v)$.

Sketch of proof:

Recall: our problem is to minimize:

$$\vec{f}^T L^T L \vec{f} \quad \text{subject to} \quad \|\vec{g} - D\vec{f}\|^2 = \varepsilon$$

$$\| \vec{g} - D\vec{f} \|^2 = (\vec{g} - D\vec{f})^T (g - D\vec{f})$$

From calculus, the minimizer must satisfy:

$$\mathcal{D} = \frac{\partial}{\partial \vec{f}} \left(\vec{f}^T L^T L \vec{f} + \lambda (\vec{g} - D\vec{f})^T (\vec{g} - D\vec{f}) \right) = 0 \quad \text{for}$$

where $\vec{f} = (f_1, f_2, \dots, f_i, \dots, f_{N^2})^T$ and λ is the Lagrange's multiplier.

Here, $\frac{\partial K}{\partial \vec{f}} = \left(\frac{\partial K}{\partial f_1}, \frac{\partial K}{\partial f_2}, \dots, \frac{\partial K}{\partial f_{N^2}} \right)^T$

Easy to check:

- $\frac{\partial(\vec{f}^T \vec{a})}{\partial \vec{f}} = \vec{a}$
- $\frac{\partial(\vec{b}^T \vec{f})}{\partial \vec{f}} = \vec{b}$
- $\frac{\partial(\vec{f}^T A \vec{f})}{\partial \vec{f}} = (A + A^T) \vec{f}$

$$\vec{f}^{T}\vec{a} = (f_1, f_2, \dots, f_n)\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 f_1 + a_2 f_2 + \dots + a_n f_n$$

$$\frac{\partial \vec{f}^{T}\vec{a}}{\partial f_j} = a_j$$

$$\therefore \frac{\partial \vec{f}^{T}\vec{a}}{\partial \vec{f}} \overset{\text{def}}{=} \left(\frac{\partial \vec{f}^{T}\vec{a}}{\partial f_1}, \dots, \frac{\partial \vec{f}^{T}\vec{a}}{\partial f_n} \right) = (a_1, a_2, \dots, a_n)$$

etc. . ..

$\therefore \quad D = 0 \Rightarrow (2L^TL)\vec{f} + \lambda(-D^T\vec{g} - D^T\vec{g} + 2D^TD\vec{f}) = 0$

$\Rightarrow (D^TD + \gamma L^TL)\vec{f} = D^T\vec{g}$ where $\gamma = \frac{1}{\lambda}$ and λ is the Lagrange's multiplier.

Parameter γ can be determined by direct substitution into the equation:

$$(\vec{g} - D\vec{f})^T(\vec{g} - D\vec{f}) = \varepsilon.$$

Now, we'll consider the frequency domain.

Note that D and L are transformation matrix of convolution.

$\therefore$ D and L are block-circulant.

Some facts about circulant matrix:

Recall: A matrix is block-circulant if

$$H = \begin{pmatrix} H_0 & H_{M-1} & H_{M-2} & \cdots & H_1 \\ H_1 & H_0 & H_{M-1} & \cdots & H_2 \\ H_2 & H_1 & H_0 & \cdots & H_3 \\ \vdots & \vdots & \vdots & & \vdots \\ H_{M-1} & H_{M-2} & H_{M-3} & \cdots & H_0 \end{pmatrix}$$

(each H_i is circulant)

A matrix $\mathcal{C}$ is circulant if:

$$\mathcal{C} = \begin{pmatrix} d_0 & d_{M-1} & d_{M-2} & \cdots & d_1 \\ d_1 & d_0 & d_{M-1} & \cdots & d_2 \\ d_2 & d_1 & d_0 & \cdots & d_3 \\ \vdots & \vdots & \vdots & & \vdots \\ d_{M-1} & d_{M-2} & d_{M-3} & \cdots & d_0 \end{pmatrix}$$

Eigenvalues / Eigenvectors of circulant $\mathcal{C}$

Let $\mathcal{C} = \begin{pmatrix} d(0) & d(M-1) & \cdots & d(1) \\ d(1) & d(0) & \cdots & d(2) \\ \vdots & \vdots & \cdots & \vdots \\ d(M-1) & d(M-2) & \cdots & d(0) \end{pmatrix}$ be a circulant matrix. Then the eigenvalues of $\mathcal{C}$ is given by:

$$\underline{\lambda(k) = d(0) + d(1)e^{\frac{2\pi j}{M}(M-1)k} + d(2)e^{\frac{2\pi j}{M}(M-2)k} + \cdots + d(M-1)e^{\frac{2\pi j}{M}k}}$$

(eigenvalue)

where $k = 0, 1, 2, \cdots M-1$.

Its associated eigenvector is given by:

$$\boxed{\vec{w}(k) = \begin{pmatrix} 1 \\ e^{\frac{2\pi j}{M}k} \\ e^{\frac{2\pi j}{M}2k} \\ \vdots \\ e^{\frac{2\pi j}{M}(M-1)k} \end{pmatrix}}$$

(eigenvector)

Diagonalization of block-circulant matrix

D (transformation matrix of $h * f$) $\in M_{N^2 \times N^2}$

Let $\cancel{H}$ D be the block-circulant matrix as defined above. Define a matrix with elements:

$$W_N(k,n) := \frac{1}{\sqrt{N}} exp\left(\frac{2\pi j}{N} kn\right) \qquad 0 \le n \le N-1$$

$\in M_{N \times N}$

$$D = W \underbrace{\Lambda_D}_{\text{diagonal}} W^{-1}$$

Consider the **Kronecker product** $\otimes$ of W_N with itself:

$$W := W_N \otimes W_N \quad \in M_{N^2 \times N^2}$$

Recall that: Kronecker product is defined as:

The **Kronecker product** of two matrices are given by:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & \cdots & a_{2N}B \\ \vdots & \vdots & & \vdots \\ a_{N1}B & a_{N2}B & \cdots & a_{NN}B \end{pmatrix}$$

$A = (a_{ij})_{0 \le i,j \le N-1}$

$B = (b_{ij})_{0 \le i,j \le N-1}$

W^{-1} can be easily computed!

Easy to check: $W^{-1} = W_N^{-1} \otimes W_N^{-1}$ where:

$$W_N^{-1}(k,n) := \frac{1}{\sqrt{N}} exp\left(-\frac{2\pi j}{N} kn\right) \qquad 0 \leq n \leq N-1$$

Let

$$\Lambda(k,i) = \begin{cases} N^2 \ddot{H}\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where H = DFT of the point spread function h, $\left\lfloor \frac{k}{N} \right\rfloor$ = largest integer smaller than or equal to $\frac{k}{N}$ and $\mathrm{mod}_N(k) = k(\mathrm{mod}\ N)$ (e.g. $10(\mathrm{mod}\ 3) = 1$)

Then, we can show that $H = W\Lambda W^{-1}$ and $H^{-1} = W\Lambda^{-1}W^{-1}$.

Also, $H^T = W\Lambda^* W^{-1}$. (Λ^* is the complex conjugate of Λ)

By direct calculation, it is easy to check that $W^{-1}\vec{g} = N\zeta(G)$ where $G = DFT(g)$.

Using the fact that both D and L are block-circulant, we can check that:

$$D = W\Lambda_D W^{-1}, D^T = W\Lambda_D^* W^{-1}, L = W\Lambda_L W^{-1}, L^T = W\Lambda_L^* W^{-1}$$

where W is invertible and Λ_D, Λ_L are diagonal matrices.

Also,

$$\Lambda_D(k,i) = \begin{cases} N^2 \underset{\substack{\| \\ DFT(h)}}{H}\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where $H = DFT(h)$.

and

$$\Lambda_L(k,i) = \begin{cases} N^2 P\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

$P = DFT(p)$;

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}$$

$$\Lambda_D = N^2 \begin{pmatrix} H(0,0) & & & & & & & \\ & H(1,0) & & & & & & \\ & & \ddots & & & & & \\ & & & H(N-1,0) & & & & \\ & & & & H(0,1) & & & \\ & & & & & H(1,1) & & \\ & & & & & & \ddots & \\ & & & & & & & H(N-1,1) & \cdots \end{pmatrix}$$

1st col of DFT(h)

2nd col of DFT(h)

Combining these information and substitute into the "governing" equation:

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g},$$

We get:

$$\cancel{W}(\Lambda_D^* \Lambda_D + \gamma \Lambda_L^* \Lambda_L) W^{-1}\vec{f} = \cancel{W}\Lambda_H^* W^{-1}\vec{g}$$

$W^{-1}\vec{f} = \begin{pmatrix} F(0,0) \\ F(1,0) \\ \vdots \\ F(N-1,0) \\ \vdots \\ F(N-1,N-1) \end{pmatrix}$, $\quad W^{-1}\vec{g} = \begin{pmatrix} G(0,0) \\ G(1,0) \\ \vdots \\ G(N-1,0) \\ \vdots \\ G(N-1,N-1) \end{pmatrix}$

Λ_D: entries given by DFT(p)

Λ_L: entries given by DFT(p)

$$D^T D = W \overline{\Lambda}_D \cancel{W^{-1}} \cancel{W} \Lambda_D W^{-1}$$

$$W \overline{\Lambda}_D \Lambda_D W^{-1}$$

$$(\overline{a} a = |a|^2)$$

$$D = W\Lambda_D W^{-1}, \quad D^T = W\Lambda_D^* W^{-1}, \quad L = W\Lambda_L W^{-1}, \quad L^T = W\Lambda_L^* W^{-1}$$

$$\Lambda_D^* = \overline{\Lambda}_D$$

Combining these information and substitute into the "governing" equation:

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g},$$

We get:

$$\cancel{W}(\Lambda_D^* \Lambda_D + \gamma \Lambda_L^* \Lambda_L) W^{-1}\vec{f} = \cancel{W}\Lambda_D^* W^{-1}\vec{g}$$

We can check that:

① $$\Lambda_D^* \Lambda_D = \begin{pmatrix} N^4|H(0,0)|^2 & & & & \\ & N^4|H(1,0)|^2 & & & \\ & & \ddots & & \\ & & & N^4|H(N-1,0)|^2 & \\ & & & & \ddots \\ & & & & & N^4|H(N-1,N-1)|^2 \end{pmatrix}$$

$H = DFT(h)$

② $$\Lambda_L^* \Lambda_L = \begin{pmatrix} N^4|P(0,0)|^2 & & & & \\ & N^4|P(1,0)|^2 & & & \\ & & \ddots & & \\ & & & N^4|P(N-1,0)|^2 & \\ & & & & \ddots \\ & & & & & N^4|P(N-1,N-1)|^2 \end{pmatrix}$$

$P = DFT(p)$

③ $\boxed{W^{-1}\vec{f} = N\mathcal{S}(F), W^{-1}\vec{g} = N\mathcal{S}(G)}$ where $F = DFT(f), G = DFT(g)$.

Combining all these, we get for every (u,v),

$$N^4[|H(u,v)|^2+\gamma|P(u,v)|^2]NF(u,v) = N^2\overline{H(u,v)}NG(u,v)$$

$$\Rightarrow \boxed{N^2\frac{|H(u,v)|^2+\gamma|P(u,v)|^2}{\overline{H(u,v)}}F(u,v) = G(u,v)} \Rightarrow F(u,v) = \frac{1}{N^2}\left(\frac{\overline{H(u,v)}}{|H(u,v)|^2+\gamma|P(u,v)|^2}\right)G(u,v)$$

Summary: Constrained least square filtering minimizes:

$$E(\vec{f}) = (L\vec{f})^T(L\vec{f})$$

subject to the constraint that:

$$\|\underbrace{\vec{g} - H\vec{f}}_{\vec{n}}\|^2 = \varepsilon$$

(allow fixed amount of noise)

Example: Assume that:

$$g = \begin{pmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix} \quad \text{and} \quad W_3^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & exp\left(-\frac{2\pi j}{3}\right) & exp\left(-\frac{2\pi j}{3}2\right) \\ 1 & exp\left(-\frac{2\pi j}{3}2\right) & exp\left(-\frac{2\pi j}{3}\right) \end{pmatrix}$$

Then:

$$W^{-1} = W_3^{-1} \otimes W_3^{-1} = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2}
\end{pmatrix}$$

$$W^{-1}\vec{g} = \frac{1}{3}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2}
\end{pmatrix}
\begin{pmatrix} g_{00} \\ g_{10} \\ g_{20} \\ g_{01} \\ g_{11} \\ g_{21} \\ g_{02} \\ g_{12} \\ g_{22} \end{pmatrix}$$

$G = DFT(g)$

$$= \frac{1}{3}\begin{pmatrix}
g_{00} + g_{10} + g_{20} + g_{01} + g_{11} + g_{21} + g_{02} + g_{12} + g_{22} \\
g_{00} + g_{10}e^{-\frac{2\pi j}{3}} + g_{20}e^{-\frac{2\pi j}{3}2} + g_{01} + g_{11}e^{-\frac{2\pi j}{3}} + g_{21}e^{-\frac{2\pi j}{3}2} + g_{02} + g_{12}e^{-\frac{2\pi j}{3}} + g_{22}e^{-\frac{2\pi j}{3}2} \\
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\begin{matrix} = 3^2\, G(0,0) \\ = 3^2\, G(1,0) \\ \vdots \\ \\ \\ \end{matrix}$$

$G = DFT(g)$

$$\therefore \quad W^{-1}\vec{g} = 3\,\mathcal{S}(G) = 3\begin{pmatrix} G(0,0) \\ G(1,0) \\ \vdots \\ G(N-1,0) \\ \vdots \\ G(N-1,N-1) \end{pmatrix}$$