MMAT5390 Mathematical Image Processing
Practice Final Examination

1. Please practice all the exercises in Chapter 1, 3, 4 and 5.

2. Consider a 3×3 periodically extended image $I = (I(k,l))_{0 \le k,l \le 2}$ given by:
$$I = \begin{pmatrix} 2a & b+2c & c \\ 3b+c & 2a+c & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
where $a, b, c \ge 0$.

   Let $h = (h(k,l))_{0 \le k,l \le 3} = \dfrac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, which is a periodically extended.

   Let $I_1(k,l) = 2I(k,l) - I(k-1,l) - I(k,l-1)$ for any $k, l \in \mathbb{Z}$.

   The Butterworth high-pass filter H of radius a and order b is defined by
$$H(u,v) = \begin{cases} \frac{1}{1+(\frac{a^2}{u^2+v^2})^b} & \text{if } (u,v) \ne (0,0) \\ 0 & \text{if } (u,v) = (0,0). \end{cases}$$
   Let $I_2(u,v) = H(u,v)DFT(I)(u,v)$.

   Suppose $h * I(1,1) = 3$, $I_1(1,2) = -4$, $DFT(I)(2,0) \ne 0$ and $I_2(2,0) = \frac{1}{2}DFT(I)(2,0)$. Find a, b, c.

3. For any periodically extended $N \times N$ image f, define
$$G_x(f)(x,y) = \frac{1}{4}f(x+1,y) + \frac{1}{2}f(x,y) + \frac{1}{4}f(x-1,y)$$
$$\text{and } G_y(f)(x,y) = \frac{1}{4}f(x,y+1) + \frac{1}{2}f(x,y) + \frac{1}{4}f(x,y-1).$$

   (a) Find an $N \times N$ image h such that for any periodically extended $N \times N$ image f,
$$h * f = G_x(G_y(f)).$$

   (b) Let $H(u,v)$ be the LPF such that
$$DFT(h * f)(u,v) = H(u,v)DFT(f)(u,v),$$
   where h is the convolution kernel from (a). Using H, perform unsharp masking (i.e. $k = 1$) on the following periodically extended 4×4 image
$$f = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

4. Suppose an atmospherically-blurred $N \times N$ image g is given by:
$$g(m,n) = \sum_{j=0}^{T} I(m - cj, n - cj),$$
where $0 \le m, n \le N-1$, c is a positive constant denoting the speed of motion and I is the original image. Assuming that I and g are periodically extended. Show that $DFT(g)(u,v) = H(u,v)DFT(I)(u,v)$, where $H(u,v)$ is the degradation function in the frequency domain. Write $H(u,v)$ in terms of c. Please show your answer with details.

5. Let $W_N(n,k) = \dfrac{1}{\sqrt{N}}e^{2\pi j \frac{nk}{N}}$ for $0 \le n, k \le N-1$ and $W = W_N \otimes W_N$.

   (a) Prove that $W^{-1} = \overline{W_N} \otimes \overline{W_N}$.

   (b) Show that $W^{-1}\mathcal{S}(f) = N\mathcal{S}(\hat{f})$ for any $f \in M_{N \times N}(\mathbb{C})$, where $\hat{f} = DFT(f)$.

6. Given $N^2 \times N^2$ block-circulant real matrices D and L, $N \times N$ image g and fixed parameter $\varepsilon > 0$, the constrained least square filtering aims to find $f \in M_{N\times N}$ that minimizes:

$$E(f) = [L\mathcal{S}(f)]^T[L\mathcal{S}(f)]$$

subject to the constraint:

$$[\mathcal{S}(g) - D\mathcal{S}(f)]^T[\mathcal{S}(g) - D\mathcal{S}(f)] = \varepsilon,$$

where $\mathcal{S}$ is the stacking operator.

(a) Prove that D is diagonalizable by $W = W_N \otimes W_N$, where $W_N(n,k) = \frac{1}{\sqrt{N}} e^{2\pi j \frac{nk}{N}}$, i.e. $W^{-1}DW$ is diagonal, and find its eigenvalues in terms of $DFT(h)$ where $D\mathcal{S}(\varphi) = \mathcal{S}(h * \varphi)$ for any $\varphi \in M_{N\times N}(\mathbb{R})$.

(b) Given that the optimal solution f that solves the constrained least square problem satisfies $[\lambda D^T D + L^T L]\mathcal{S}(f) = \lambda D^T \mathcal{S}(g)$ for some parameter λ. Find $DFT(f)$ in terms of $DFT(g)$, $DFT(h)$, $DFT(p)$ and λ, where $L\mathcal{S}(\varphi) = \mathcal{S}(p * \varphi)$ for any $\varphi \in M_{N\times N}(\mathbb{R})$.

7. Consider the following curve evolution model for image segmentation. Let $\gamma_t := \gamma_t(s) : [0, 2\pi] \to D$ be a closed curve in the image domain D. We proceed to find γ that minimizes:

$$E_{snake,2}(\gamma) = \int_0^{2\pi} \frac{1}{2}\|\gamma'(s)\|^2\, ds + \alpha \int_0^{2\pi} \frac{1}{2}\|\gamma''(s)\|^2\, ds + \beta \int_0^{2\pi} V(\gamma(s))\, ds,$$

where V is the edge detector, α and β are fixed positive parameters. Assume that γ' and γ'' are discretized by:

$$\gamma'(s_i) = [\gamma(s_{i+1}) - \gamma(s_i)]/\sigma \text{ and}$$
$$\gamma''(s_i) = [\gamma(s_{i+1}) - 2\gamma(s_i) + \gamma(s_{i-1})]/\sigma^2.$$

(a) Derive the gradient descent iterative scheme to minimize $E_{snake,2}$ in the continuous setting.

(b) Discretize $E_{snake,2}$.

(c) Derive the explicit Euler scheme (using gradient descent method) to iteratively minimize the discrete version of $E_{snake,2}$.

8. Using the gradient descent algorithm, the active contour model iteratively minimizes:

$$E_{snake}(\mathbf{u}) = \sum_{i=1}^{N} \frac{1}{2}\left|\frac{\mathbf{u}_{i+1} - \mathbf{u}_i}{\sigma}\right|^2 + \beta \sum_{i=1}^{N} V(\mathbf{u}_i)$$

where $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_N)^T \in M_{N\times 2}(\mathbb{R})$ is a discrete closed curve ($\mathbf{u}_i \in \mathbb{R}^2$ for all i), $V(x,y) = x^2 + y^2$ is the edge detector, β is a fixed positive constant and $\sigma = \frac{2\pi}{N}$ is the arc-length parameter.

(a) The explicit Euler scheme of active contour model is given by:

$$\frac{\mathbf{u}_i^{k+1} - \mathbf{u}_i^k}{\tau} = \frac{\mathbf{u}_{i+1}^k - 2\mathbf{u}_i^k + \mathbf{u}_{i-1}^k}{\sigma^2} - \beta\nabla V(\mathbf{u}_i^k) \text{ for } k = 0, 1, 2, ...$$

where τ is the time step. Let $\mathbf{u}^k = (\mathbf{u}_1^k, \mathbf{u}_2^k, ..., \mathbf{u}_N^k)^T \in M_{N\times 2}(\mathbb{R})$. Prove that $E_{snake}(\mathbf{u}^{k+1}) \leq E_{snake}(\mathbf{u}^k)$ for $k = 0, 1, 2, ...$ if τ is small enough.