MMAT5390: Mathematical Image Processing

Assignment 4

Due: April 25 2022 before 12pm

Please give reasons in your solutions. You can use the hints directly **without** proving them.

1. Given $N^2 \times N^2$ block-circulant real matrices D and L, $N \times N$ image g and fixed parameter $\varepsilon > 0$, the constrained least square filtering aims to find $f \in M_{N\times N}$ that minimizes:

$$E(f) = [L\mathcal{S}(f)]^T [L\mathcal{S}(f)]$$

subject to the constraint:

$$[\mathcal{S}(g) - D\mathcal{S}(f)]^T [\mathcal{S}(g) - D\mathcal{S}(f)] = \varepsilon,$$

where $\mathcal{S}$ is the stacking operator.

   (a) Prove that D is diagonalizable by $W = W_N \otimes W_N$, where $W_N(n,k) = \frac{1}{\sqrt{N}} e^{2\pi j \frac{nk}{N}}$, i.e. $W^{-1}DW$ is diagonal, and find its eigenvalues in terms of $DFT(h)$ where $D\mathcal{S}(\varphi) = \mathcal{S}(h * \varphi)$ for any $\varphi \in M_{N\times N}(\mathbb{R})$. (**hints**: You may use $W^{-1} = \overline{W_N} \otimes \overline{W_N}$ and $D(x,y) = h(mod_N(x) - mod_N(y), \lfloor \frac{x}{N} \rfloor - \lfloor \frac{y}{N} \rfloor)$.)

   (b) Given that the optimal solution f that solves the constrained least square problem satisfies $[\lambda D^T D + L^T L]\mathcal{S}(f) = \lambda D^T \mathcal{S}(g)$ for some parameter λ. Find $DFT(f)$ in terms of $DFT(g)$, $DFT(h)$, $DFT(p)$ and λ, where $L\mathcal{S}(\varphi) = \mathcal{S}(p * \varphi)$ for any $\varphi \in M_{N\times N}(\mathbb{R})$. (**hints**: You may use $W^{-1}\mathcal{S}(f) = N\mathcal{S}(\hat{f})$, where $\hat{f} = DFT(f)$.)

2. Verify that for any $N \times N$ circulant matrix

$$C = \begin{pmatrix} c_0 & c_{N-1} & c_{N-2} & \cdots & c_1 \\ c_1 & c_0 & c_{N-1} & \cdots & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{N-1} & c_{N-2} & c_{N-3} & \cdots & c_0 \end{pmatrix},$$

$UC\bar{U}$ is diagonal, where U is the $N \times N$ DFT matrix $U = (U(x,\alpha))_{0 \le x,\alpha \le N-1}$ with $U(x,\alpha) = \frac{1}{N} e^{-2\pi j \frac{x\alpha}{N}}$. Thus find the eigenvalues of C in terms of $c_0, c_1, \ldots, c_{N-1}$.

3. Given a noisy image $g(x,y)$, we consider the image denoising algorithm to obtain a clean image $f(x,y)$ through minimizing the following energy functional:

$$E(f) = \int_\Omega \{|f(x,y) - g(x,y)|^2 + \lambda \|\nabla f(x,y)\|^2\}\, dx\, dy$$

where λ is a constant parameter. Derive an iterative scheme to minimize $E(f)$ in the continuous setting.

4. Consider the following curve evolution model for image segmentation. Let $\gamma_t := \gamma_t(s) : [0, 2\pi] \to D$ be a closed curve in the image domain D. We proceed to find γ that minimizes:

$$E_{snake,2}(\gamma) = \int_0^{2\pi} \frac{1}{2}\|\gamma'(s)\|^2\, ds + \alpha \int_0^{2\pi} \frac{1}{2}\|\gamma''(s)\|^2\, ds + \beta \int_0^{2\pi} V(\gamma(s))\, ds,$$

where V is the edge detector, α and β are fixed positive parameters.

Derive the gradient descent iterative scheme to minimize $E_{snake,2}$ in the continuous setting.