MATH3360: Mathematical Imaging

Assignment 2

1. (a) $H_0(t) = \mathbf{1}_{[0,1)}$, and for any $p \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{Z} \cap [0, 2^p - 1]$,

$$H_{2^p+n}(t) = 2^{\frac{p}{2}} \left(\mathbf{1}_{[\frac{n}{2^p}, \frac{n+0.5}{2^p})} - \mathbf{1}_{[\frac{n+0.5}{2^p}, \frac{n+1}{2^p})} \right).$$

The Haar transform matrix $\tilde{H} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$.

(b)

$$\begin{aligned}
A_{\text{Haar}} &= \tilde{H}^T A \tilde{H} \\
&= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 3 & 9 & 0 \\ 3 & 9 & 0 & 5 \\ 9 & 0 & 5 & 3 \\ 0 & 5 & 3 & 9 \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 17 & 17 & 17 & 17 \\ -1 & 7 & -1 & 7 \\ 2\sqrt{2} & -6\sqrt{2} & 9\sqrt{2} & -5\sqrt{2} \\ 9\sqrt{2} & -5\sqrt{2} & 2\sqrt{2} & -6\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 17 & 0 & 0 & 0 \\ 0 & 3 & -2\sqrt{2} & 2\sqrt{2} \\ 0 & -2\sqrt{2} & 4 & 7 \\ 0 & 2\sqrt{2} & 7 & 4 \end{pmatrix}.
\end{aligned}$$

(c) The modified Haar transform $\tilde{A}_{\text{Haar}}$ is $\begin{pmatrix} 17 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 7 & 4 \end{pmatrix}$, and thus:

$$\begin{aligned}
\tilde{A} &= \tilde{H}^T \tilde{A}_{\text{Haar}} \tilde{H} \\
&= \frac{1}{4} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 17 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 7 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 17 & 3 & 4\sqrt{2} & 7\sqrt{2} \\ 17 & 3 & -4\sqrt{2} & -7\sqrt{2} \\ 17 & -3 & 7\sqrt{2} & 4\sqrt{2} \\ 17 & -3 & -7\sqrt{2} & -4\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 7 & 3 & 7 & 0 \\ 3 & 7 & 0 & 7 \\ 7 & 0 & 7 & 3 \\ 0 & 7 & 3 & 7 \end{pmatrix}.
\end{aligned}$$

2. (a) $W_0 = \mathbf{1}_{[0,1)}$, and for any $j \in \mathbb{N} \cup \{0\}$ and $q \in \{0, 1\}$,

$$W_{2j+q}(t) = (-1)^{\lfloor \frac{j}{2} \rfloor + q} (W_j(2t) + (-1)^{j+q} W_j(2t-1)).$$

The Walsh transform matrix $\tilde{W} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$.

(b)

$$\begin{aligned} B_{\text{Walsh}} &= \tilde{W} B \tilde{W}^T \\ &= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 7 & 8 \\ 3 & 7 & 6 & 2 \\ 2 & 6 & 3 & 5 \\ 2 & 8 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 8 & 24 & 20 & 16 \\ 0 & 4 & -6 & -4 \\ 2 & 2 & -2 & -2 \\ -2 & -6 & 0 & 10 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 17 & 1 & 5 & -3 \\ -1.5 & -3.5 & 0.5 & -1.5 \\ 0 & -2 & 0 & 0 \\ 0.5 & 4.5 & -3.5 & -1.5 \end{pmatrix}. \end{aligned}$$

(c) The modified Walsh transform $\tilde{B}_{\text{Walsh}}$ is $\begin{pmatrix} 1 & 1 & 5 & -3 \\ -1.5 & -3.5 & 0.5 & -1.5 \\ 0 & -2 & 0 & 0 \\ 0.5 & 4.5 & -3.5 & -1.5 \end{pmatrix}$, and thus:

$$\begin{aligned} \tilde{B} &= \tilde{W}^T \tilde{B}_{\text{Walsh}} \tilde{W} \\ &= \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 5 & -3 \\ -1.5 & -3.5 & 0.5 & -1.5 \\ 0 & -2 & 0 & 0 \\ 0.5 & 4.5 & -3.5 & -1.5 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 3 & 11 & 1 & -3 \\ 2 & -2 & 8 & 0 \\ 0 & 0 & 2 & -6 \\ -1 & -5 & 9 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -3 & -1 & 3 & 4 \\ -1 & 3 & 2 & -2 \\ -2 & 2 & -1 & 1 \\ -2 & 4 & 0 & -3 \end{pmatrix}. \end{aligned}$$

3. (a) $\int_{\mathbb{R}} [H_0(t)]^2\, dt = \int_0^1 dt = 1$.

For any $p \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{Z} \cap [0, 2^p - 1]$,

$$\begin{aligned} \int_{\mathbb{R}} [H_{2^p+n}(t)]^2\, dt &= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} (2^{\frac{p}{2}})^2\, dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}} (-2^{\frac{p}{2}})^2\, dt \\ &= 2 \cdot \frac{1}{2^{p+1}} \cdot 2^p = 1. \end{aligned}$$

(b) i. Let $m \in \mathbb{N} \setminus \{0\}$. There exists $p \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{Z} \cap [0, 2^p - 1]$ such that $m = 2^p + n$. Then

$$\begin{aligned} \langle H_0, H_m \rangle &= \int_{\mathbb{R}} H_0(t) H_{2^p+n}(t)\, dt \\ &= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} 2^{\frac{p}{2}}\, dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}} (-2^{\frac{p}{2}})\, dt \\ &= \frac{1}{2^{p+1}} \cdot 2^{\frac{p}{2}} + \frac{1}{2^{p+1}} \cdot (-2^{\frac{p}{2}}) = 0. \end{aligned}$$

ii. A. Suppose $p_1 = p_2$. Then

$$\begin{aligned}\langle H_{m_1}, H_{m_2}\rangle &= \int_{\mathbb{R}} H_{2^{p_1}+n_1}(t) H_{2^{p_1}+n_2}(t)\, dt \\ &= \int_{\frac{n_1}{2^{p_1}}}^{\frac{n_1+0.5}{2^{p_1}}} 2^{\frac{p_1}{2}} \cdot 0\, dt + \int_{\frac{n_1+0.5}{2^{p_1}}}^{\frac{n+1}{2^{p_1}}} (-2^{\frac{p_1}{2}}) \cdot 0\, dt \\ &+ \int_{\frac{n_2}{2^{p_1}}}^{\frac{n_2+0.5}{2^{p_1}}} 0 \cdot 2^{\frac{p_1}{2}} + \int_{\frac{n_2+0.5}{2^{p_1}}}^{\frac{n_2+1}{2^{p_1}}} 0 \cdot (-2^{\frac{p_1}{2}})\, dt = 0.\end{aligned}$$

B. Suppose $p_1 < p_2$, $p_1, p_2 \in \mathbb{N} \cup \{0\}$. Then $p_1 + 1 \le p_2$, $p_2 - p_1 \ge 1$. The length of $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right)$ is equal to $\frac{1}{2^{p_2}}$

- The length of $\left[0, \frac{n_1}{2^{p_1}}\right)$ is $\frac{n_1}{2^{p_1}}$. Since $\frac{n_1}{2^{p_1}}/\frac{1}{2^{p_2}} = n_1 2^{p_2-p_1}$, the length of $\left[0, \frac{n_1}{2^{p_1}}\right)$ is a multiple of that of $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right)$.
- The length of $\left[\frac{n_1}{2^{p_1}}, \frac{n_1+0.5}{2^{p_1}}\right)$ is $\frac{1}{2^{p_1+1}}$. Since $\frac{1}{2^{p_1+1}}/\frac{1}{2^{p_2}} = 2^{p_2-p_1-1}$, the length of $\left[\frac{n_1}{2^{p_1}}, \frac{n_1+0.5}{2^{p_1}}\right)$ is a multiple of that of $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right)$.
- The length of $\left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right)$ is $\frac{1}{2^{p_1+1}}$. Since $\frac{1}{2^{p_1+1}}/\frac{1}{2^{p_2}} = 2^{p_2-p_1-1}$, the length of $\left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right)$ is a multiple of that of $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right)$.
- Considering the possible subset relations between the supports of H_{m_1} and H_{m_2}, we notice that
  - $2^{p_2-p_1}n_1 \le n_2 < 2^{p_2-p_1}(n_1+0.5)$ and thus $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \subseteq \left[\frac{n_1}{2^{p_1}}, \frac{n_1+0.5}{2^{p_1}}\right)$; or
  - $2^{p_2-p_1}(n_1+0.5) \le n_2 < 2^{p_2-p_1}(n_1+1)$ and thus $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \subseteq \left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right)$; or
  - $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \cap \left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right) = \varnothing$.

  In any case, H_{m_1} is constant on $[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}})$, and thus denoting the constant by c,

$$\begin{aligned}\langle H_{m_1}, H_{m_2}\rangle &= \int_{\mathbb{R}} H_{2^{p_1}+n_1}(t) H_{2^{p_2}+n_2}(t)\, dt \\ &= c\int_{\frac{n_2}{2^{p_2}}}^{\frac{n_2+0.5}{2^{p_2}}} 2^{\frac{p_2}{2}}\, dt + c\int_{\frac{n_2+0.5}{2^{p_2}}}^{\frac{n_2+1}{2^{p_2}}} (-2^{\frac{p_2}{2}})\, dt \\ &= c[\frac{1}{2^{p_2+1}} \cdot 2^{\frac{p_2}{2}} + \frac{1}{2^{p_2+1}} \cdot (-2^{\frac{p_2}{2}})] = 0.\end{aligned}$$

4. (a) The 2D discrete Fourier transform (DFT) of an $M \times N$ image $g = (g(k,l))_{k,l}$, where $k = 0, 1, \cdots, M-1$ and $l = 0, 1, \cdots, N-1$, is defined as:

$$\hat{g}(m,n) = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l) e^{-2\pi j(\frac{km}{M}+\frac{ln}{N})}$$

The Fourier transform matrix $U = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$.

(b)

$$
\begin{aligned}
C_{\mathrm{DFT}} &= UCU \\
&= \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}\begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\
&= \begin{pmatrix} 2.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.5-0.5j & 0.5 & 0.5+0.5j \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

(c) The modified Fourier coefficient matrix

$$
\tilde{C}_{\mathrm{DFT}} = \begin{pmatrix} 2.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.5-0.5j & 0 & 0.5+0.5j \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

Thus,

$$
\begin{aligned}
\tilde{C} &= (4U^*)\tilde{C}_{\mathrm{DFT}}(4U^*) \\
&= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}\begin{pmatrix} 2.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.5-0.5j & 0 & 0.5+0.5j \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \\
&= \begin{pmatrix} 3.5 & 3.5 & 1.5 & 1.5 \\ 1.5 & 1.5 & 3.5 & 3.5 \\ 3.5 & 3.5 & 1.5 & 1.5 \\ 1.5 & 1.5 & 3.5 & 3.5 \end{pmatrix}
\end{aligned}
$$

5. **Coding Assignment:**

Q1:

MATLAB:

```
    recon = (h * U') * freq * (h * U');
```

Python:

```
    recon = (h * U.T.conjugate()) @ freq @ (h * U.T.conjugate())
```