MATH5390: Mathematical Imaging

Assignment 3

1. (a) The matrix U used to calculate the DFT of an $N \times N$ matrix is given by

$$U = (U(x,\alpha))_{0 \le x,\alpha \le n}, \text{ where } U(x,\alpha) = \frac{1}{\sqrt{N}} e^{2\pi j \frac{\alpha x}{N}}.$$

To check that U is unitary, we first denote the column of U indexed by α by $\vec{u}_\alpha$. Then,

i. For any $0 \le \alpha \le N-1$,

$$\begin{aligned}
\langle \vec{u}_\alpha, \vec{u}_\alpha \rangle &= \sum_{x=0}^{N-1} U(x,\alpha)\overline{U(x,\alpha)} \\
&= \sum_{x=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi j \frac{x\alpha}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{x\alpha}{N}} \\
&= N \cdot \frac{1}{N} = 1.
\end{aligned}$$

ii. For any $0 \le \alpha_1, \alpha_2 \le N_1$ such that $\alpha_1 \ne \alpha_2$,

$$\begin{aligned}
\langle \vec{u}_{\alpha_1}, \vec{u}_{\alpha_2} \rangle &= \sum_{x=0}^{N-1} U(x,\alpha_1)\overline{U(x,\alpha_2)} \\
&= \sum_{x=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi j \frac{x\alpha_1}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{x\alpha_2}{N}} \\
&= \frac{1}{N} \sum_{x=0}^{N-1} e^{2\pi j \frac{x(\alpha_1 - \alpha_2)}{N}} \\
&= \frac{1}{N} \cdot N\mathbf{1}_{N\mathbb{Z}}(\alpha_1 - \alpha_2) = 0.
\end{aligned}$$

Hence U is unitary.

(b) For any $0 \le p, q \le N-1$,

$$\begin{aligned}
iDFT(DFT(f))(p,q) &= \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k,l) e^{2\pi j \frac{m(k-p)+n(l-q)}{N}} \\
&= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k,l) \left[\sum_{m=0}^{N-1} e^{2\pi j \frac{m(k-p)}{N}} \right] \left[\sum_{n=0}^{N-1} e^{2\pi j \frac{n(l-q)}{N}} \right] \\
&= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k,l) \cdot N\mathbf{1}_{N\mathbb{Z}}(k-p) \cdot N\mathbf{1}_{N\mathbb{Z}}(l-q) \\
&= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k,l)\delta(k-p)\delta(l-q) = f(p,q).
\end{aligned}$$

2. There are two different approaches.

Direct proof

$$\widehat{f \odot g}(k,l) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)g(m,n)e^{-2\pi j (\frac{mk}{M} + \frac{nl}{N})}.$$

On the other hand,

$$\hat{f} * \hat{g}(k,l) = \sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1} \hat{f}(a,b)\hat{g}(k-k', l-l')$$

$$= \frac{1}{M^2N^2}\sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1}\left(\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{-2\pi j(\frac{mk'}{M}+\frac{nl'}{N})}\right)$$

$$\left(\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} g(m',n')e^{-2\pi j(\frac{m'(k-k')}{M}+\frac{n'(l-l')}{N})}\right)$$

$$= \frac{1}{M^2N^2}\sum_{k',m,m'=0}^{M-1}\sum_{l',n,n'=0}^{N-1} f(m,n)g(m',n')e^{-2\pi j(\frac{k'(m-m')+m'k}{M}+\frac{l'(n-n')+n'l}{N})}$$

$$= \frac{1}{M^2N^2}\sum_{m,m'=0}^{M-1}\sum_{n,n'=0}^{N-1} f(m,n)g(m',n')e^{-2\pi j(\frac{m'k}{M}+\frac{n'l}{N})}M\delta(m-m')N\delta(n-n')$$

$$= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)g(m,n)e^{-2\pi j(\frac{mk}{M}+\frac{nl}{N})} = \widehat{f \odot g}(k,l).$$

Indirect proof (via inverse transform)

$$iDFT(\hat{f} * \hat{g})(m,n) = \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} \hat{f} * \hat{g}(k,l)e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})}$$

$$= \sum_{k,k'=0}^{M-1}\sum_{l,l'=0}^{N-1} \hat{f}(k',l')\hat{g}(k-k',l-l')e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})}$$

$$= \sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1}\sum_{k''=k-M+1}^{k}\sum_{l''=l-N+1}^{l} \hat{f}(k',l')\hat{g}(k'',l'')e^{2\pi j(\frac{m(k'+k'')}{M}+\frac{n(l'+l'')}{N})}$$

$$= \left[\sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1} \hat{f}(k',l')e^{2\pi j(\frac{mk'}{M}+\frac{nl'}{N})}\right]\left[\sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1} \hat{g}(k'',l'')e^{2\pi j(\frac{mk''}{M}+\frac{nl''}{N})}\right]$$

$$= f(m,n)g(m,n).$$

3. (a) The matrix T used to calculate the EDCT of an $N \times N$ image is given by

$$T = (T(x,\alpha))_{0\le x,\alpha\le N-1}, \text{ where } T(x,\alpha) = \frac{1}{N}\cos\frac{\pi x(\alpha+\frac{1}{2})}{N}.$$

(b) Let $N > 1$. Denote by $\vec{T_x}$ the row of T indexed by x.
Suppose cT is unitary for some $c \in \mathbb{R}$.
Then

$$1 = \langle c\vec{T_0}, c\vec{T_0}\rangle = \frac{c^2}{N^2}\sum_{\alpha=0}^{N-1}\cos^2\frac{\pi\cdot 0(\alpha+\frac{1}{2})}{N} = \frac{c^2}{N},$$

and thus $c^2 = N$.

On the other hand,

$$
\begin{aligned}
1 &= \langle c\vec{T}_1, c\vec{T}_1\rangle \\
&= \frac{c^2}{N^2}\sum_{\alpha=0}^{N-1}\cos^2\frac{\pi\cdot 1(\alpha+\frac{1}{2})}{N} \\
&= \frac{c^2}{2N^2}\sum_{\alpha=0}^{N-1}[1+\cos(2\pi\frac{2\alpha+1}{2N})] \\
&= \frac{c^2}{2N}+\frac{c^2}{4N^2}\sum_{\alpha=-N}^{N-1}\cos(2\pi\frac{1\cdot(2\alpha+1)}{2N}) \\
&= \frac{c^2}{2N}+\frac{c^2}{2N}\mathbf{1}_{2N\mathbb{Z}}(1)=\frac{c^2}{2N}=\frac{1}{2}.
\end{aligned}
$$

Contradiction. Hence cT is not unitary for any $c\in\mathbb{R}$.

4. The given information implies

$$\sqrt{e}(e^{-\frac{10}{\sigma^2}})=e^{-\frac{5}{\sigma^2}},$$

$$e^{\frac{1}{2}-\frac{10}{\sigma^2}}=e^{-\frac{5}{\sigma^2}}.$$

Hence $\sigma^2-20=-10$,
and thus $\sigma^2=10$.

5. The given information implies

$$H(1,0)=\frac{64}{65}\text{ and }H(-1,1)=\frac{8}{9}.$$

Hence

$$\begin{cases}\dfrac{D_0^{2n}}{D_0^{2n}+1^n} &= \dfrac{64}{65},\\ \dfrac{D_0^{2n}}{D_0^{2n}+2^n} &= \dfrac{8}{9},\end{cases}\text{ and thus }\begin{cases}64(D_0^{2n}+1) &= 65D_0^{2n},\\ 8(D_0^{2n}+2^n) &= 9D_0^{2n}.\end{cases}$$

Then $D_0^{2n}=64=8\times 2^n$. Hence $n=3$ and $D_0=2$.

6. **Coding Assignment:**

MATLAB:

```
filter = exp(-dist/2/(sigma^2));
```

Python:

```
low_pass_filter = np.exp(-dist/2/(sigma**2))
```