Lecture 3 : Revision (3)

Linear Transformation $T: V \to W$

① $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}) \quad \forall \vec{x}, \vec{y} \in V$

② $T(a\vec{x}) = a\,T(\vec{x}) \quad \forall a \in \mathbb{F}, \forall \vec{x} \in V$

Examples of linear transformations :

(1) Identity transformation : $I_V : V \to V$ by $I_V(\vec{x}) = \vec{x}$

(2) Zero transformation : $T_0 : V \to W$ by $T_0(\vec{x}) = \vec{0}_W \quad \forall \vec{x} \in V$

(3) Let $A \in M_{m\times n}$. $L_A : \mathbb{F}^n \to \mathbb{F}^m$ defined by $L_A(\vec{x}) = A\vec{x}$.

Given $T: V \to W$, two important subspaces :

kernel / null space : $N(T) := \{\vec{x} \in V : T(\vec{x}) = \vec{0}\} \subseteq V$

range / image space : $R(T) := \{T(\vec{x}) : \vec{x} \in V\} \subseteq W$

Important fact :

- T is 1-1 $\Leftrightarrow$ $N(T) = \{\vec{0}\}$
- T is onto $\Leftrightarrow$ $R(T) = W$
- T is called an isomorphism if T is 1-1 and onto
- (Dimension Thm) $\underbrace{\text{Nullity}(T)}_{= \dim(N(T))} + \underbrace{\text{rank}(T)}_{= \dim(R(T))} = \dim(V)$

  (Refer to P.70)
- If $\dim(V) = \dim(W)$, then:

  T is 1-1 $\Leftrightarrow$ T is onto $\Leftrightarrow$ $\text{rank}(T) = \dim(V)$
- Matrix representation of $T: V \to W$

  Let β and γ be bases of V and W respectively.

  Say $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

  Then, the matrix representation of T is given by:

$$[T]_\beta^\gamma = \begin{pmatrix} | & | & & | \\ [T(\vec{v}_1)]_\gamma & [T(\vec{v}_2)]_\gamma & \cdots & [T(\vec{v}_n)]_\gamma \\ | & | & & | \end{pmatrix} \in M_{m\times n}$$

(Assume $m = \dim(W)$, $n = \dim(V)$)

- We have the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \beta\ \wr\!\!\updownarrow & & \updownarrow\!\wr\ \gamma \\ \mathbb{F}^n & \xrightarrow{L_A} & \mathbb{F}^m \end{array}$$

where $A = [T]_\beta^\gamma$, $L_A : \mathbb{F}^n \to \mathbb{F}^m$ is defined by: $L_A(\vec{x}) = A\vec{x}$.

Hence, we have: $[T(\vec{v})]_\gamma = [T]_\beta^\gamma [\vec{v}]_\beta$

$\therefore$ a linear transformation T can be identified by a matrix $A := [T]_\beta^\gamma$ (given bases β and γ)

- Let $T : V \to W$; $S : U \to V$

Let α, β, γ be the bases of U, V, W resp.

Then: $$[T \circ S]_\alpha^\gamma = [T]_\beta^\gamma [S]_\alpha^\beta$$

(Refer to P.88)

(Relationship between composition of transformations and matrix multiplication)

- Change of coordinate matrix

Consider $T : V \to V$. Let β and γ be two different bases of V.

Then: $\underset{\substack{\| \\ \text{simple notation} \\ \text{for } [T]_\beta^\beta}}{[T]_\beta}$ is similar to $[T]_\gamma$

(A is similar to B if $\exists$ invertible Q s.t.

$$B = Q^{-1} A Q \;)$$

In fact, $[T]_\gamma = Q^{-1} [T]_\beta \, Q$ where

$$Q = [I_V]_\gamma^\beta$$

(Reason: $\underset{\gamma}{V} \xrightarrow[I_V]{} \underset{\beta}{V} \xrightarrow{T} \underset{\beta}{V} \xrightarrow[I_V]{} \underset{\gamma}{V}$

$$[T]_\gamma = [\, I_V \circ T \circ I_V \,]_\gamma = \underbrace{[\, I_V]_\beta^\gamma}_{Q^{-1}} \, [T]_\beta \, \underbrace{[I_V]_\gamma^\beta}_{Q}$$

Easy to check that $[I_V]_\beta^\gamma$ is the inverse of $[I_V]_\gamma^\beta$)

<u>Example of change of basis</u>

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto x-axis:

$$T(a_1, a_2) = (a_1, 0)$$

Let $\beta = \{ \overset{e_1 =}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \overset{= e_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \}$ be the standard basis.

Then: $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \cdot \vec{e}_1 + 0 \, \vec{e}_2$

$$T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \vec{e}_1 + 0 \, \vec{e}_2$$

$\therefore \left[T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\left[T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Thus, $[T]_\beta = \left([T(\vec{e}_1)]_\beta \;\; [T(\vec{e}_2)]_\beta \right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Now, consider another basis of $\mathbb{R}^2$:

$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \therefore \left[T\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{\beta'} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$

$T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \therefore \left[T\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]_{\beta'} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

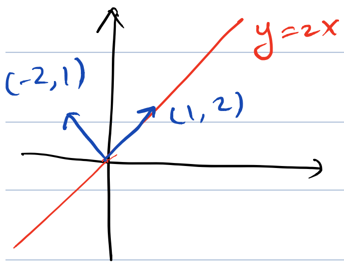
$\therefore \ [T]_{\beta'} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$

Now, $[I_V]_{\beta'}^{\beta} = \left(\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{\beta} \ \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]_{\beta} \right) = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{Q}$

Easy to check that: $\underset{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}}{\underset{\parallel}{[T]_{\beta'}}} = Q^{-1} \underset{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}{\underset{\parallel}{[T]_{\beta}}} Q$

Example: (Reflection)

Find the linear transformation T, which is the reflection about the line $y=2x$.



Let $\beta' = \Big\{ \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\vec{u}_1}, \underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{\vec{u}_2} \Big\}$

If $\vec{v} = \lambda_1 \vec{u}_1 + \lambda_2 \vec{u}_2$

then $T(\vec{v}) = \lambda_1 \vec{u}_1 - \lambda_2 \vec{u}_2$

$$\therefore \quad [T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

With respect to standard ordered basis $\beta = \Big\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big\}$

$$[T]_\beta = \underbrace{[I_V]^{\beta}_{\beta'}}_{Q} \, [T]_{\beta'} \, \underbrace{[I_V]^{\beta'}_{\beta}}_{Q^{-1}}$$

$$Q = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \quad ; \quad Q^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$\therefore \quad [T]_\beta = Q[T]_{\beta'}Q^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$$

$$\therefore \quad T\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{-3a+4b}{5} \\ \frac{4a+3b}{5} \end{pmatrix}$$

Isomorphism (P. 99 - P. 105)

Definition: Let $T: V \to W$ (linear). T is called invertible if there exists $U: W \to V$ (linear) such that $U \circ T = I_V$ and $T \circ U = I_W$. (I_V and I_W are identity maps)

Remark:

- The inverse of T is unique. We denote it by T^{-1}.
- $(T^{-1})^{-1} = T$. Hence, T^{-1} is also invertible.
- $(TU)^{-1} = U^{-1} \circ T^{-1}$
- Let $T: V \to W$. Both V and W are finite-dimensional. Then:

  T is invertible $\Rightarrow$ Nullity$(T) = 0$ and rank$(T) = \dim(V)$

  (T is invertible $\Rightarrow$ T is 1-1 and onto
  $\Rightarrow$ rank$(T) = \dim(V)$
  (by dimension Thm))

- An invertible linear transformation $T: V \to W$ is also called an isomorphism.
- V and W are called isomorphic to each others if $\exists$ isomorphism $T: V \to W$.

Example 1: Let $T: P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by:

$$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = \begin{pmatrix} a_0 + a_1, & a_0 - a_1 \\ a_2 + a_3, & a_2 - a_3 \end{pmatrix}$$

Define $U: M_{2\times 2}(\mathbb{R}) \to P_3(\mathbb{R})$ by:

$$U\left(\underbrace{\begin{pmatrix} y_0 & y_1 \\ y_2 & y_3 \end{pmatrix}}_{\in M_{2\times 2}(\mathbb{R})}\right) = \left(\frac{y_0+y_1}{2}\right) + \left(\frac{y_0-y_1}{2}\right)x + \left(\frac{y_2+y_3}{2}\right)x^2 + \left(\frac{y_2-y_3}{2}\right)x^3 \in P_3(\mathbb{R})$$

Then: $T^{-1} = U$.

Example 2: Let $T: P_5(\mathbb{R}) \to P_4(\mathbb{R})$ by $T(f(x)) = f'(x)$

Let $U: P_4(\mathbb{R}) \to P_5(\mathbb{R})$ by $U(f(x)) = \int_0^x f(t)\,dt$

We can check that $TU = I_{P_4(\mathbb{R})}$.

BUT: $UT \neq I_{P_5(\mathbb{R})}$

(Consider $UT(f(x) + c) = U(f'(x)) = f(x) \neq f(x) + c$)

$c \neq 0$

with $f(0) = 0$

$\therefore$ U is not the inverse of T.

More properties of isomorphisms

- Let $T: V \to W$ be invertible linear transformation. Then: $\dim(V) < \infty$ iff $\dim(W) < \infty$. Also, $\boxed{\dim(V) = \dim(W)}$

- $T: V \to W$. V, W = finite dimensional.
  β and γ are ordered bases of V and W respectively.
  Then: T is invertible iff $[T]_\beta^\gamma$ is invertible.
  (matrix)

  Also, $[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$.

- V, W = finite-dim vector spaces over the same field
  V isomorphic to W iff $\dim(V) = \dim(W)$.

  (Let $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ = ordered basis of V.
  $\gamma = \{\vec{w}_1, \ldots, \vec{w}_n\}$ = " " " W.

  Define $T(\vec{v}_i) = \vec{w}_i$ for $i = 1, 2, \ldots, n$.

  Then: $T: V \to W$ is an isomorphism.)

  Remark: Let $\dim(V) = n$. Then: V isomorphic to $\mathbb{F}^n$.

  This is how we define V "equal" to $\mathbb{F}^n$.

Important Theorem: Let V and W be finite dimensional vector spaces over $\mathbb{F}$ of dimension n and m respectively. Let β and γ be ordered bases of V and W respectively.
Then: $\Phi: \mathcal{L}(V, W) \to M_{m \times n}(\mathbb{F})$ defined by:
("collection" of all lin transf. from V to W)

$\Phi(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V, W)$ is an isomorphism.

Proof: Easy to check that: $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ and
$[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$. Hence, Φ is linear.

To show that Φ is an isomorphism, we need to prove that Φ is 1-1 and onto.

Onto: Given $A \in M_{m\times n}(\mathbb{F})$, we need to show $\exists T \in \mathcal{L}(V,W)$ such that $\Phi(T) = A$.

Let $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $\gamma = \{\vec{w}_1, \dots, \vec{w}_m\}$.

Recall: A linear transformation is determined by its values at the basis elements.

Also, the matrix representation of T w.r.t. β and γ is given by (A_{ij}) where $T(\vec{v}_j) = \sum_{i=1}^m A_{ij}\vec{w}_i$

So, $\exists T: V \to W$ such that $T(\vec{v}_j) = \sum_{i=1}^m A_{ij}\vec{w}_i$ for $1 \le j \le n$.

Then: $[T]_\beta^\gamma = A$. Hence, $\Phi(T) = A$.

1-1: $[T]_\beta^\gamma = [U]_\beta^\gamma \Rightarrow T = U$ (Why?)

Remark: $\mathcal{L}(V,W)$ isomorphic to $M_{m\times n}(\mathbb{F})$
$\Downarrow$
Lin. transf. "=" Matrice $\Rightarrow$ Study of T equiv. Study of $[T]_\beta^\gamma$.