Lecture 25 : Jordan Canonical Form (Part 3)

Theoretical Proof: Assume $T: V \to V$ (fin-dim) and char poly splits.
$\lambda_1, \lambda_2, \dots, \lambda_k$ = distinct eigenvalues.

First question: $V = K_{\lambda_1} \oplus \dots \oplus K_{\lambda_k}$

Claim 1: $T - \lambda_i I\big|_{K_{\lambda_j}} : K_{\lambda_j} \to K_{\lambda_j}$ is 1-1. if $i \neq j$.

Pf: Let $\vec{x} \in K_{\lambda_j}$ and $(T - \lambda_i I)(\vec{x}) = \vec{0}$. Suppose $\vec{x} \neq \vec{0}$.
Let p = smallest integer such that $(T - \lambda_j I)^p(\vec{x}) = \vec{0}$.
Let $\vec{y} = \underbrace{(T - \lambda_j I)^{p-1}(\vec{x})}_{\neq \vec{0}}$. Then: $(T - \lambda_j I)\vec{y} = (T - \lambda_j I)^p(\vec{x}) = \vec{0}$.
$\therefore \vec{y} \in E_{\lambda_j}$. Now, $(T - \lambda_i I)(\vec{y}) = (T - \lambda_i I)(T - \lambda_j I)^{p-1}(\vec{x})$
$= (T - \lambda_j I)^{p-1}\underbrace{(T - \lambda_i I)(\vec{x})}_{\vec{0}} = \vec{0}$
$\therefore \vec{y} \in E_{\lambda_i}$. But $E_{\lambda_i} \cap E_{\lambda_j} = \{\vec{0}\}$.
$\therefore \vec{y} = \vec{0}$. Contradiction. $\therefore \vec{x} = \vec{0}$ and so $(T - \lambda_i I)\big|_{K_{\lambda_j}}$ is 1-1.

Claim 2: $\dim(K_{\lambda_i}) \leq m_i$ = multiplicity of λ_i and
$K_\lambda = N((T - \lambda_i I)^{m_i})$.

Pf: ① Let $g(t)$ = char poly of $T|_{K_{\lambda_i}}$. Then $g(t)$ divides char poly of T. Now, $(T - \lambda_j I)\big|_{K_{\lambda_i}}(\vec{x}) \neq \vec{0}$ if $\lambda_j \neq \lambda_i$ and $\vec{x} \neq \vec{0}$.
$\therefore \lambda_i$ is the only eigenvalue.
Hence, $g(t) = (-1)^d (t - \lambda_i)^d$, $d = \dim(K_{\lambda_i})$. $\therefore$ $d \leq m_i$
$\Rightarrow \dim(K_{\lambda_i}) \leq m_i$.

② $N((T-\lambda_i I)^{m_i}) \subseteq K_{\lambda_i}$ by definition.

Now, $g(T|_{K_{\lambda_i}}) = 0$ by Caley-Hamilton Thm.

$\therefore (T|_{K_{\lambda_i}} - \lambda_i I)^d = 0 \Rightarrow (T-\lambda_i I)^d(\vec{x}) = \vec{0}$ for $\forall \vec{x} \in K_{\lambda_i}$

$\Rightarrow (T-\lambda_i I)^{m_i}(\vec{x}) = \vec{0}$ for $\forall \vec{x} \in K_{\lambda_i}$

Hence, $K_{\lambda_i} \subseteq N((T-\lambda_i I)^{m_i})$.

Claim 3: $V = K_{\lambda_1} + K_{\lambda_2} + \dots + K_{\lambda_k}$

Pf: By M.I. on k = # of distinct eigenvalues.

When $k=1$, let m = multiplicity of λ_1. Then, char poly of $T = (\lambda_1 - t)^m$.

By Caley-Hamilton Thm, $g(T) = (\lambda_1 I - T)^m = T_0 \leftarrow$ zero transf.

$\therefore K_{\lambda_1} = N((T-\lambda_1 I)^m) = V$. Thm is true for $k=1$.

Assume the thm is true for transf. w/ fewer than k eigenvalues.

We'll show thm is true for k distinct eigenvalues.

Claim 4: Let $W = R((T-\lambda_k I)^m)$ $\leftarrow$ multiplicity of λ_k

Then: $T|_W : W \to W$ is well-defined and $T|_W$ has $k-1$ distinct eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_{k-1}$.

Besides, $(T-\lambda_k I)^m|_{K_{\lambda_i}} : K_{\lambda_i} \to K_{\lambda_i}$ is onto $(i<k)$

Assume Claim 4 is true. Let $\vec{x} \in V$. Then: $(T-\lambda_k I)^m \vec{x} \in W$.

By induction hypothesis, $\exists \vec{w}_i \in K'_{\lambda_i}$ = generalized eigenspace of λ_i of $T|_W$.

Such that: $(T-\lambda_k I)^m \vec{x} = \vec{w}_1 + \vec{w}_2 + \dots + \vec{w}_{k-1}$.

Easy to check that: $K'_{\lambda_i} \subseteq K_{\lambda_i}$ for $i<k$.

Since $(T-\lambda_k I)^m|_{K_{\lambda_i}}: K_{\lambda_i} \to K_{\lambda_i}$ is onto, for each $\vec{w}_i \in K'_{\lambda_i}$,

$\exists \vec{v}_i \in V \ni (T-\lambda_k I)^m(\vec{v}_i) = \vec{w}_i$

$\therefore (T-\lambda_k I)^m(\vec{x}) = (T-\lambda_k I)^m(\vec{v}_1) + \dots + (T-\lambda_k I)^m(\vec{v}_{k-1})$

$\Leftrightarrow (T-\lambda_k I)^m(\vec{x} - \vec{v}_1 - \dots - \vec{v}_{k-1}) = \vec{0}$.

$\therefore \vec{x} - \vec{v}_1 - \dots - \vec{v}_{k-1} \in N((T-\lambda_k I)^m) = K_{\lambda_k}$

$\therefore \vec{x} = \vec{v}_1 + \dots + \vec{v}_{k-1} + \underset{\in K_{\lambda_k}}{\vec{v}_k}$

By M.I., the thm is true.

Proof of Claim 4:

Let $W = R((T-\lambda_k I)^m)$. Since T and $(T-\lambda_k I)^m$ commute, W is T-invariant and so $T|_W$ is well-defined.

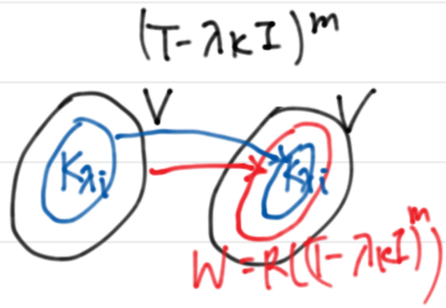
Consider $(T-\lambda_k I)^m|_{K_{\lambda_i}}: K_{\lambda_i} \to K_{\lambda_i}$ for $i<k$ (K_{λ_i} is $(T-\lambda_k I)^m$-invariant)

We'll prove: $(T-\lambda_k I)^m|_{K_{\lambda_i}}$ is onto.

Note that: $(T-\lambda_k I)|_{K_{\lambda_i}}$ is 1-1 and hence onto.

So, $(T-\lambda_k I)^m|_{K_{\lambda_i}}$ is also onto.

Also, $E_{\lambda_i} \subseteq K_{\lambda_i} \subseteq W = R((T-\lambda_k I)^m)$ because:



$\therefore \lambda_i$ is an eigenvalue of $T|_{K_{\lambda_i}}$

$\therefore \lambda_1, \lambda_2, \dots, \lambda_{k-1}$ are eigenvalues of $T|_W$.

Next, suppose λ_k is an eigenvalue of $T|_W$.

Suppose $T|_W(\vec{v}) = \lambda_k \vec{v}$ for $\vec{v} \neq \vec{0} \in W = R((T-\lambda_k I)^m)$

Then: $\vec{v} = (T-\lambda_k I)^m(\vec{y})$, $\vec{y} \in V$.

Thus, $\vec{0} = (T-\lambda_k I)\vec{v} = (T-\lambda_k I)^{m+1}(\vec{y})$.

So, $\vec{y} \in K_{\lambda_k} = N((T-\lambda_k I)^m)$ and so $(T-\lambda_k I)^m(\vec{y}) = \vec{v} = \vec{0}$

$\because$ eigenvalues of $T|_W \Rightarrow$ eigenvalues of T. (Contradiction)

$\therefore T|_W$ has exactly $k-1$ distinct eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_{k-1}$.

Claim 5: Let β_i = ordered basis of K_{λ_i}. Then:

$\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$ is a disjoint union and a basis of V.

Pf: Disjoint union: Let $x \in \beta_i \cap \beta_j$ $(i \neq j) \subseteq K_{\lambda_i} \cap K_{\lambda_j}$.

Since $(T-\lambda_i I)|_{K_{\lambda_j}}$ is 1-1, we get: $(T-\lambda_i I)(x) \neq \vec{0}$

$(T-\lambda_i I)^p(x) \neq \vec{0}$ for all positive p.

Contradiction since $x \in K_{\lambda_i}$. $\therefore \beta_i \cap \beta_j = \phi$

Now, let $x \in V$. By Claim 3, $x = v_1 + v_2 + \dots + v_k$ where $v_i \in K_{\lambda_i}$.

$\therefore x$ is a lin. comb. of vectors in β. $\therefore V = \text{span}(\beta)$

Let $q = |\beta|$. Then, $\dim(V) \leq q$.

Let $d_i = \dim(K_{\lambda_i})$. Then: $q = \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} m_i = \dim(V)$.

$\therefore q = \dim(V) \Rightarrow \beta$ = basis of V.

Claim 6: $\dim(K_{\lambda_i}) = m_i$

Pf: $\sum_{i=1}^{k} d_i = \sum_{i=1}^{k} m_i \Rightarrow \sum_{i=1}^{k}(m_i - d_i) = 0$.

$\Rightarrow m_i = d_i \ \forall i$ ($\because m_i \geq d_i$)

Claim 7: Let $\gamma_1 = \{(T-\lambda_1 I)^{m_1}\vec{v}_1, \ldots, \vec{v}_1\}$

$\vdots$

$\gamma_q = \{(T-\lambda I)^{m_q}\vec{v}_q, \ldots, \vec{v}_q\}$

lin ind.

If initial vectors are distinct, then $\gamma = \gamma_1 \cup \ldots \cup \gamma_q$ is disjoint union and it is lin. ind.

Pf: Disjoint union: HW.

Lin ind: Use M.I on $n = \#$ of elements in γ.

When $n=1$, trivial.

Assume the thm is true when γ has less than n elements.

When $|\gamma| = n$, let

$\gamma_1' = \{(T-\lambda I)^{m_1}\vec{v}_1, \ldots, (T-\lambda I)\vec{v}_1\}$

$\vdots$

$\gamma_q' = \{(T-\lambda I)^{m_q}\vec{v}_q, \ldots, (T-\lambda I)\vec{v}_q\}$

Let $\gamma' = \gamma_1' \cup \ldots \cup \gamma_q'$. Let $W = \mathrm{span}(\gamma)$. Let $U = (T-\lambda I)|_W$.

Then: $R(U) = \mathrm{span}(\gamma')$ (Check)

Also, $|\gamma'| = n-q$. Since initial vectors of γ_i's are distinct,

γ' is lin. ind. Thus, $\dim(R(U)) = n-q$.

Also, $S = \{(T-\lambda I)^{m_1}\vec{v}_1, \ldots, (T-\lambda I)^{m_q}(\vec{v}_q)\} \subseteq N(U)$

$\therefore \dim(N(U)) \geq q$

$\therefore n \geq \dim(W) = \dim(R(U)) + \dim(N(U)) = (n-q) + q = n$

$\therefore \dim(W) = n$, $|\gamma| = n$ and $\mathrm{span}(\gamma) = W$.

$\therefore$ γ is a basis of W and γ is lin. ind.