Lecture 20: More about unitary and orthogonal operators

Corollary 1: Let $T: V \to V$ be a linear operator on a finite-dimensional real inner product space.
Then: V has an o.n. basis of eigenvectors of T with corresponding eigenvalues of absolute value 1 if and only if T is both self-adjoint and orthogonal.

Corollary 2: Let $T: V \to V$ be a linear operator on a finite dimensional complex inner product space.
Then: V has an o.n. basis of eigenvectors of T with corresponding eigenvalues of absolute value 1 if and only if T is unitary.

Proof of Corollary 1:

($\Rightarrow$) Suppose V has an o.n. basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ such that $T(\vec{v}_i) = \lambda_i \vec{v}_i$ and $|\lambda_i| = 1$ for all i.
Then, by thm, T is self-adjoint ($T^* = T$)
Thus, $(TT^*)(\vec{v}_i) = T(T(\vec{v}_i)) = \lambda_i^2 \vec{v}_i$
$\therefore$ $TT^* = I$ and hence $TT^* = T^*T = I$ (orthogonal)

($\Leftarrow$) Suppose T is self-adjoint. By thm, $\exists$ o.n. basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ such that $T(\vec{v}_i) = \lambda_i \vec{v}_i$ for all i.
As T is orthogonal, $\|T(\vec{v}_i)\| = \|\vec{v}_i\|$
$\therefore$ $|\lambda_i| \|\vec{v}_i\| = \|\lambda_i \vec{v}_i\| = \|T(\vec{v}_i)\| = \|\vec{v}_i\|$
$\because$ $\|\vec{v}_i\| \neq 0$ $\therefore$ $|\lambda_i| = 1$ for all i.

Remark: Proof of Corollary 2 is similar.

Definition: A square matrix $A \in M_{n\times n}(\mathbb{F})$ is called an <u>orthogonal matrix</u> if $A^TA = AA^T = I$.

A is called a <u>unitary matrix</u> if $A^*A = AA^* = I$.

Remark:

- For real matrix $A \in M_{n\times n}(\mathbb{R})$, an unitary matrix implies orthogonal matrix (since $A^* = A^T$) Hence, we call A orthogonal rather than unitary.
- $AA^* = I \iff$ Rows of A forms orthonormal basis because:

$$\delta_{ij} = I_{ij} = (AA^*)_{ij} = \sum_{k=1}^{n} A_{ik} (A^*)_{kj}$$

$$= \underbrace{\sum_{k=1}^{n} A_{ik} \overline{A_{jk}}}_{\substack{\text{inner product of} \\ (A_{i1}, A_{i2}, \ldots, A_{in}) \text{ and} \\ (A_{j1}, A_{j2}, \ldots, A_{jn})}}$$

- Similarly, $A^*A = I \iff$ Cols of A forms an o.n. basis.
- T on an inner product space V is unitary (orthogonal)

$$\Updownarrow$$

$[T]_\beta$ is unitary (orthogonal) for some o.n. basis β for V.

Example: (Orthogonal operator: reflection)

Definition: Let L be a 1-dim subspace of $\mathbb{R}^2$. (L = line in a plane through the origin). $T: \mathbb{R}^2 \to \mathbb{R}^2$ is called a reflection of $\mathbb{R}^2$ about L if $T(\vec{x}) = \vec{x}$ for $\forall \vec{x} \in L$ and $T(\vec{x}) = -\vec{x}$ if $\vec{x} \in L^{\perp}$.

Let $\vec{v}_1 \in L$ and $\vec{v}_2 \in L^{\perp}$ such that $\|\vec{v}_1\| = \|\vec{v}_2\| = 1$.

Then: $T(\vec{v}_1) = \vec{v}_1$ and $T(\vec{v}_2) = -\vec{v}_2$.

Thus, $\vec{v}_1$, $\vec{v}_2$ are eigenvectors of T with eigenvalue equal to 1 or -1.

Also, $\{\vec{v}_1, \vec{v}_2\}$ forms an orthonormal basis of $\mathbb{R}^2$.

Hence, T must be self-adjoint and orthogonal.

Unitarily (orthogonally) equivalent

Observation: If A is a complex normal matrix, then $\exists$ o.n. basis consisting of eigenvectors of A.

Hence, $D = Q^{-1} A Q$ for some diagonal matrix D.

Also, Q is a matrix whose columns form an orthonormal set. Hence, Q is unitary ($Q^* Q = Q Q^* = I$).

Hence, $D = Q^* A Q$.

Similarly, if A is symmetric (self-adjoint), then:

$$D' = Q^T A Q \text{ for some diagonal matrix } D'.$$

Definition: Let $A, B \in M_{n \times n}(\mathbb{C})$ ($M_{n \times n}(\mathbb{R})$)

We say that A and B are unitarily (orthogonally) equivalent if and only if there exists a unitary (orthogonal) matrix P such that $A = P^* B P$ ($A = P^T B P$)