Lecture 16: More about orthogonal complement and adjoint of linear operators

Recall: Suppose W is a subspace of an inner product space V. Then:

$$W^{\perp} \overset{\text{def}}{:=} \{\vec{x} \in V : \langle \vec{x}, \vec{y} \rangle = 0 \text{ for all } \vec{y} \in W\}$$

Theorem 1: Suppose $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ = orthonormal subset in n-dim inner product space V. Then:

① S can be extended to orthonormal basis of V: $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$

② If $W = \text{span}(S)$, then $S_1 = \{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ = orthonormal basis for $W^{\perp}$.

③ If W is any subspace of V, then: $\dim(V) = \dim(W) + \dim(W^{\perp})$

Proof: **Lemma:** If $\{\vec{w}_1, \dots, \vec{w}_n\}$ = orthogonal set of non-zero vectors, then $\{\vec{v}_1, \dots, \vec{v}_n\}$ derived from G-S process satisfy:

$\vec{v}_i = \vec{w}_i$ for $\forall i$.

① S can be extended to an ordered basis $S' = \{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_{k+1}, \dots, \vec{w}_n\}$ for V. Apply G-S process to S'. The first k vectors remain the same according to the lemma. Normalize S' to get an orthonormal basis S_1 for V: $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\} := \beta$

② S_1 is linearly independent ($\because S_1$ = subset of a basis)

Now, $S_1 = \{\vec{v}_{k+1}, \ldots, \vec{v}_n\} \subseteq W^\perp$ (obvious)

We need to prove $W^\perp = \text{span}(S_1)$

Note that for $\forall \vec{x} \in V$, we have:

$$\vec{x} = \sum_{i=1}^{n} \langle \vec{x}, \vec{v}_i \rangle \vec{v}_i$$

If $\vec{x} \in W^\perp$, then $\langle \vec{x}, \vec{v}_i \rangle = 0$ for $i = 1, 2, \ldots, k$

So, $\vec{x} = \sum_{i=k+1}^{n} \langle \vec{x}, \vec{v}_i \rangle \vec{v}_i \in \text{Span}(S_1)$

By construction, S_1 is orthonormal (hence lin. ind)

$\therefore S_1$ = orthonormal basis for $W^\perp$

③ $\dim V = |\beta| = |S| + |S_1| = k + (n-k)$
$= \dim(W) + \dim(W^\perp)$.

Example 1: Let $W = \text{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_k\} \subseteq \mathbb{R}^n$
Then: $W^\perp = \text{span}\{e_{k+1}, \ldots, e_n\} \subseteq \mathbb{R}^n$

$$\therefore \dim(V = \mathbb{R}^n) = n = \underset{\overset{\shortparallel}{k}}{\dim(W)} + \underset{\overset{\shortparallel}{n-k}}{\dim(W^\perp)}$$

Adjoint of a linear operator

Recall: Adjoint/conjugate transpose A^*

Goal: • Define adjoint T^* of T such that $([T]_\beta)^* = [T^*]_\beta$

• Conjugate of complex number $\leftrightarrow$ adjoint of T.

Observation: V = inner product space, $\vec{y} \in V$. Define $g: V \to \mathbb{F}$ by $g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$. g is linear functional.

Fact: If V = fin-dim, then every linear functional f can be written as: $f(x) = \langle \vec{x}, \vec{y} \rangle$ for some $\vec{y} \in V$.

Theorem 2: V = fin-dim inner product space and $g: V \to \mathbb{F}$ be linear functional. Then: $\exists$ unique $\vec{y} \in V$ such that $g(x) = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x} \in V$.

Proof: Let $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ = orthonormal basis for V.

Let $\vec{y} = \sum_{i=1}^{n} \overline{g(\vec{v}_i)}\, \vec{v}_i$

Define $h: V \to \mathbb{F}$ by $h(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ (linear)

Want to prove: $g = h$. ($\Leftrightarrow g(\vec{v}_j) = h(\vec{v}_j)$ for $\forall j$)

$$h(\vec{v}_j) = \langle \vec{v}_j, \sum_{i=1}^{n} \overline{g(\vec{v}_i)}\, \vec{v}_i \rangle = \sum_{i=1}^{n} g(\vec{v}_i) \langle \vec{v}_j, \vec{v}_i \rangle = g(\vec{v}_j) \underbrace{\langle \vec{v}_j, \vec{v}_j \rangle}_{1}$$

$\therefore g = h$. Now, we show $\vec{y}$ is unique.

Suppose $g(\vec{x}) = \langle \vec{x}, \vec{y}' \rangle$ for all $\vec{x}$.

Then: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{y}' \rangle$ for all $\vec{x} \Rightarrow \vec{y} = \vec{y}'$ (from previous theorem)