Lecture 12: Inner product space (Definition)

Goal: Give the concept of measurement (magnitude, length, angle, orthogonality / "perpendicular") for vectors.

Definition 1: Let V = vector space over $\mathbb{F}$. An inner product on V is a function assigning every ordered pair of vectors $\vec{x}, \vec{y} \in V$ a scalar in $\mathbb{F}$, denote it by $\langle \vec{x}, \vec{y} \rangle$, with the following properties: for $\forall \vec{x}, \vec{y}, \vec{z} \in V$, $\forall c \in \mathbb{F}$,

(a) $\langle \vec{x} + \vec{z}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$

(b) $\langle c\vec{x}, \vec{y} \rangle = c \langle \vec{x}, \vec{y} \rangle$

(c) $\overline{\langle \vec{x}, \vec{y} \rangle} = \langle \vec{y}, \vec{x} \rangle$ ($\overline{\cdot}$ = complex conjugation)

(d) $\langle \vec{x}, \vec{x} \rangle > 0$, if $\vec{x} \neq \vec{0}$.

Remark:
- If $\mathbb{F} = \mathbb{R}$, (c) simply means $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$
- If $\vec{y}, \vec{v}_1, \dots, \vec{v}_n \in V$, $a_1, a_2, \dots, a_n \in \mathbb{F}$, then:

  (i) $\langle \sum_{i=1}^n a_i \vec{v}_i, \vec{y} \rangle = \sum_{i=1}^n a_i \langle \vec{v}_i, \vec{y} \rangle$

  (ii) $$\langle \vec{y}, \sum_{i=1}^n a_i \vec{v}_i \rangle = \overline{\langle \sum_{i=1}^n a_i \vec{v}_i, \vec{y} \rangle} = \overline{\sum_{i=1}^n a_i \langle \vec{v}_i, \vec{y} \rangle}$$
  $$= \sum_{i=1}^n \overline{a_i}\, \overline{\langle \vec{v}_i, \vec{y} \rangle}$$
  $$= \sum_{i=1}^n \overline{a_i} \langle \vec{y}, \vec{v}_i \rangle$$

Example 1: $\mathbb{C}^n$: Let $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$, $\vec{y} = (y_1, \ldots, y_n) \in \mathbb{C}^n$.

Define: $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$ (Check (a), (b), (c), (d) are valid)

e.g. $\vec{x} = (1+i, 2+i)$, $\vec{y} = (3, i)$

Then: $\langle \vec{x}, \vec{y} \rangle = (1+i) \cdot \overline{3} + (2+i)\overline{i} = 3+3i - 2i + 1$
$= 4 + 2i \in \mathbb{C}$.

Example 2: $V = C([0,1], \mathbb{R})$. Let $f, g \in V$. Define:

$$\langle f, g \rangle = \int_0^1 f(t) g(t) dt.$$ (a), (b), (c) are obviously valid.

For (d), if $f(t) \neq 0$, then $f(t) \neq 0$ on some interval $I \subset [0,1]$. (using continuity of f). So, $\langle f, f \rangle = \int_0^1 f(t)^2 dt \geq \int_I f(t)^2 > 0$.

Example 3: Let $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$
$\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$

Define: $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} r x_i y_i$

- If $r > 0$, $\langle \cdot, \cdot \rangle$ defines an inner product.
- If $r < 0$, $\langle \cdot, \cdot \rangle$ is NOT an inner product.
  For $\vec{x} \neq \vec{0}$, $\langle \vec{x}, \vec{x} \rangle = \sum_{i=1}^{n} r x_i^2 = r \sum_{i=1}^{n} x_i^2 < 0$ (Contradiction to (d))