Lecture 5:

Last time: If $a_0, a_1, \ldots, a_N, b_1, \ldots, b_N$ are chosen in such a way that they minimize:

$$E(a_0, \ldots, a_N, b_1, \ldots, b_N) = \int_0^{2\pi} \left(f(x) - \sum_{k=0}^{N} a_k \cos kx + b_k \sin kx \right)^2 dx$$

Then:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx \;; \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos kx \, dx \;; \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin kx \, dx$$

Fourier coefficient.

Question: Convergence of the Fourier Series to $f(x)$?

We state without proof:

Theorem: Let $f(x)$ be a 2π-periodic real smooth function.

Let $f_N(x)$ be its truncated Fourier Series:

$$f_N(x) = \sum_{k=0}^{N} (a_k \cos kx + b_k \sin kx)$$

Then: $$f(x) = \lim_{N \to \infty} f_N(x) = \sum_{k=0}^{\infty} (a_k \cos kx + b_k \sin kx)$$

Remark: Converge for smooth 2π-periodic function.

Example: 1. Real Fourier Series of $f(x) = \sin^2 x$:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underset{\sin^2 x}{\underset{\shortparallel}{f(x)}}\, dx = \frac{1}{2}, \quad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\frac{(1-\cos 2x)}{2}}{\underset{\shortparallel}{\underset{\sin^2 x}{\underset{\shortparallel}{f(x)}}}} \cos kx\, dx = \begin{cases} 0 & k \neq 2 \\ -\frac{1}{2} & k = 2 \end{cases}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\frac{(1-\cos 2x)}{2}}{\underset{\shortparallel}{f(x)}} \sin kx\, dx = 0$$

$$\therefore\ f(x) = \frac{1}{2} - \frac{1}{2}\cos 2x$$ (Well-known trigonometric formula)

2. Real Fourier Series of $f(x) = x$:

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right) \quad \text{for } -\pi < x < \pi$$

$$\left(a_0 = a_1 = a_2 = \cdots = 0;\ b_k = (-1)^{k+1}\frac{2}{k}\right)$$

<u>Example</u>: Suppose $f(x) = 1$ on $[0, \pi]$.

If $f(x)$ is extended to $[-\pi, \pi]$ as an even function : $f(x) = \begin{cases} 1 & x \in [0,\pi] \\ 1 & x \in [-\pi, 0) \end{cases}$

Then, $a_0 = 1$, $a_k = b_k = 0$ for $(k \neq 0)$.

$\therefore$ Real Fourier Series of $f(x)$ is $\underline{1}$ (Recovering the original function)

If $f(x)$ is extended to $[-\pi, \pi]$ as an odd function: $f(x) = \begin{cases} 1 & x \in [0,\pi] \\ -1 & x \in [-\pi, 0) \end{cases}$

Then: $a_0 = 0$, $a_k = 0$, $(k \neq 0)$ $b_k = \begin{cases} 0 & k \text{ is even} \\ \frac{4}{k\pi} & k \text{ is odd} \end{cases}$

$\therefore$ Real Fourier Series of $f(x)$:

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right) \text{ for } x \in [-\pi, \pi]$$

Relationship between real and complex Fourier Series

Recall: $e^{ikx} = \cos kx + i \sin kx$; $e^{-ikx} = \cos kx - i \sin kx$.

Let $f(x) \in V$. Then: $C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx \quad (k>0)$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cos kx\, dx - \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x) \sin kx\, dx$$

$$= \frac{1}{2} a_k - \frac{i}{2} b_k$$

Similarly, $C_{-k} = \frac{1}{2}(a_k + i b_k)$ and $C_0 = a_0$.

Thus, $(k>0)$ $C_k = \overline{C_{-k}}$. (Fourier coefficients are repeated)

Also, $a_k = C_k + C_{-k}$ etc...

Quick overview of (Fourier) Spectral method

Idea: Transform the differential equation from "spatial" domain to "spectral" domain. ($\rightarrow$ algebraic equation)

e.g. Poisson eqt: $\dfrac{d^2u}{dx^2} = g(x)$ (u and g are 2π-periodic)

Suppose the Fourier (complex) coefficients of $g(x)$ is given by:

$$\ldots, \hat{g}(-k), \ldots, \hat{g}(-1), \hat{g}(0), \hat{g}(1), \ldots, \hat{g}(k), \ldots$$

Assume the Fourier (complex) coefficients of $u(x)$ is given by:

$$\ldots, \hat{u}(-k), \ldots, \hat{u}(-1), \hat{u}(0), \hat{u}(1), \ldots, \hat{u}(k), \ldots$$

Then, we have

$$\sum_{k=-\infty}^{\infty} \hat{u}(k)(+ik)^2 e^{ikx} = \sum_{k=-\infty}^{\infty} \hat{g}(k) e^{ikx} \Rightarrow -k^2 \hat{u}(k) = \hat{g}(k)$$

(Algebraic eqt)

In discrete case, a differential equation can be discretized as:

$$D\vec{u} = \vec{g}$$

where $\vec{u} = \begin{pmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_N) \end{pmatrix}$ = values of u at N points $\{x_1, x_2, \ldots, x_N\}$

$\vec{g} = \begin{pmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{pmatrix}$ = values of g at N points $\{x_1, x_2, \ldots, x_N\}$

D = $N \times N$ matrice approximating the differential operator.

<u>Question:</u> Can we "transform" $\vec{u}$ and $\vec{g}$ to turn the (BIG) linear system to SIMPLE algebraic equation?

<u>Answer:</u> YES! Discrete Fourier Transform!!

Before describing discrete Fourier Transform, we introduce (continuous) Fourier Transform:

<u>Definition:</u> For a given smooth $f(x)$ defined on $(-\infty, \infty)$, the Fourier transform of f is a function $\hat{f}$ depending on the frequency k:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \quad , \quad -\infty < k < \infty$$

The inverse Fourier Transform of $\hat{f}(k)$ recovers the original function $f(x)$:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk \quad , \quad -\infty < x < \infty$$

<u>Remark:</u> The Fourier Transform handles functions defined on $(-\infty, \infty)$ (not just $[-\pi, \pi]$)

Example: Let $f(x) = \begin{cases} 1 & \text{if } |x| \leq a \\ 0 & \text{if } |x| > a \end{cases}$

Then: $\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx = \int_{-a}^{a} 1\, \underbrace{e^{-ikx}}_{\cos kx + i \sin kx}\, dx = \dfrac{2 \sin ka}{k}$

(Note that $\hat{f}(k) \to 2a$ as $k \to 0$)

Remark: You may use the fact that $\int e^{ikx} = \frac{1}{ik} e^{ikx}$ (Check)

Example: For $a > 0$, let $f(x) = \begin{cases} e^{-ax} & x \geq 0 \\ -e^{ax} & x < 0. \end{cases}$

Then: $\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx = \int_0^{\infty} e^{-ax-ikx}\, dx + \int_{-\infty}^{0} -e^{ax-ikx}\, dx$

$$= -\frac{1}{(a+ik)} e^{-ax-ikx}\Big|_0^{\infty} - \frac{1}{a-ik} e^{ax-ikx}\Big|_{-\infty}^{0} = \frac{1}{a+ik} - \frac{1}{a-ik} = -\frac{2ik}{a^2+k^2}$$

Important properties of Fourier Transform:

① $\widehat{\alpha f}(k) = \alpha \hat{f}(k)$ for all $\alpha \in \mathbb{R}$

② $\widehat{f+g}(k) = \hat{f}(k) + \hat{g}(k)$

③ $\widehat{\frac{df}{dx}}(k) = (ik)\hat{f}(k)$

Proof: $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk \Rightarrow \frac{df}{dx}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (ik)\hat{f}(k) e^{ikx} dk$

Also, $\frac{df}{dx}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\frac{df}{dx}}(k) e^{ikx} dk$

Comparing the two formula, we get: $\widehat{\frac{df}{dx}}(k) = (ik)\hat{f}(k)$.

Go back to solving differential equation:

<u>Example:</u> Consider the ODE: $\frac{dy}{dt} - 4y = H(t)e^{-4t}$ where $t \in \mathbb{R}$

and $H(t)$ is given by: $H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0. \end{cases}$ (Heaviside function)

Apply Fourier Transform on both sides:

$$\widehat{\frac{dy}{dt}}(k) - 4\hat{y}(k) = \widehat{H(t)e^{-4t}}(k)$$

$$(ik)\hat{y}(k) - 4\hat{y}(k) = \int_{-\infty}^{\infty} H(t)e^{-4t}e^{-ikt}\,dt$$

Kill differential operator

$$= \int_0^{\infty} e^{-(4+ik)t}\,dt = \frac{1}{4+ik}$$

$$\Rightarrow (ik-4)\hat{y}(k) = \frac{1}{4+ik} \Rightarrow \hat{y}(k) = -\frac{1}{16+k^2}$$

Easy to check: $\widehat{e^{-a|t|}}(k) = \dfrac{2a}{a^2+k^2}$.

$\therefore \widehat{\frac{1}{8}e^{-4|t|}}(k) = \dfrac{1}{16+k^2}$.

$\therefore y(t) = -\dfrac{e^{-4|t|}}{8}$.

<u>Remark</u>: 1. Fourier Transform "kills" the differential operator and turns differential equation into an algebraic equation.

2. Fourier Transform extend previous "Fourier Series" spectral method to handle functions defined on the whole real axis.

Example: We consider: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ where $c \neq 0$

($u = u(x,t)$ is a function defined on $-\infty < x < \infty$, $t \geq 0$)

Solution: Apply Fourier transform with respect to x:

LHS : $\displaystyle\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} e^{-ikx} \, dx = \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} u(x,t) \, e^{-ikx} \, dx$

$$= \frac{\partial^2}{\partial t^2} \hat{u}(k,t)$$

RHS : $c^2 (ik)^2 \hat{u}(k,t) = -c^2 k^2 \hat{u}(k,t)$

$\therefore$ We have: $\dfrac{\partial^2}{\partial t^2} \hat{u}(k,t) = -c^2 k^2 \hat{u}(k,t)$ for $-\infty < k < \infty$

Using the method introduced before, we note that:

e^{ikct} and e^{-ikct} are eigenfunctions with eigenvalue $= -c^2k^2$.

$\therefore$ we conclude that: $\hat{u}(k,t) = \hat{F}(k)e^{-ikct} + \hat{G}(k)e^{ikct}$.

Apply the inverse Fourier Transform:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,t)\, e^{ikx}\, dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(k) e^{ik(x-ct)}\, dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}(k)\, e^{ik(x+ct)}\, dk$$

$$= F(x-ct) + G(x+ct)$$

$\therefore$ $u(x,t) = F(x-ct) + G(x+ct)$ for some functions F and G.