Lecture 18:

Inverse power method with shift

Goal: Take $\mu \in \mathbb{R}$. Find the eigenvalue of A closest to μ.

Observation: Consider $B = A - \mu I$. Then B has eigenvalues:

$$\{\lambda_1 - \mu, \lambda_2 - \mu, \ldots, \lambda_n - \mu\} \leftarrow$$

Inverse Power method find eigenvalues such that $|\lambda_j - \mu|$ is the smallest.

$\therefore$ λ_j closest to μ can be found.

Algorithm: (Inverse power method with shift)

Step 1: Take $\mu \in \mathbb{R}$. Pick $\vec{x}^{(0)}$ such that $\|\vec{x}^{(0)}\|_\infty = 1$.

Step 2: For $k = 1, 2, \ldots$

Solve: $(A - \mu I)\vec{w} = \vec{x}^{(k-1)}$ for $\vec{w}$.

Let: $\vec{x}^{(k)} = \dfrac{\vec{w}}{\|\vec{w}\|_\infty}$.

Let $\rho_k = \|A\vec{x}^{(k)}\|_\infty$ ($\rho_k \to |\lambda_j|$ as $k \to \infty$)

Convergence rate:

$|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$

1. Power method:

   Converges if $\eta \overset{\text{def}}{=} \left|\frac{\lambda_2}{\lambda_1}\right| < 1$ and $\langle \vec{v}_1, \vec{x}^{(0)} \rangle \neq 0$ ($\vec{v}_1$ = eigenvector of λ_1)

   Also, $\rho_k = \| A\vec{x}^{(k)} \|_\infty = |\lambda_1 + O(\eta^k)|$ (Slow convergence if $\eta \approx 1$!)

2. Inverse Power method:

   Converges if $\left|\frac{1/\lambda_{n-1}}{1/\lambda_n}\right| = \left|\frac{\lambda_n}{\lambda_{n-1}}\right| < 1$ and $\langle \vec{v}_n, \vec{x}^{(0)} \rangle \neq 0$ ($\vec{v}_n$ = eigenvector of λ_n)

   Also, $\rho_k = \| A\vec{x}^{(k)} \|_\infty = |\lambda_n + O(\eta^k)|$. (Slow convergence if $\eta \approx 1$!)

3. Inverse Power method with shift, let λ_j be closest to μ.

   Converges if: $\eta = \max_{m \neq j} \left|\frac{\lambda_j - \mu}{\lambda_m - \mu}\right| < 1$ and $\langle \vec{v}_j, \vec{x}^{(0)} \rangle \neq 0$ ($\vec{v}_j$ = eigenvector of λ_j.)

   $\rho_k = \| A\vec{x}^{(k)} \|_\infty = |\lambda_j + O(\eta^k)|$ (Slow convergence if $\eta \approx 1$!)

How to speed up convergence? Let $A \in M_{n\times n}(\mathbb{R})$

Idea: Use Inverse Power method with shift, update μ in each iteration (such that μ is closer to a real eigenvalue in each iteration)

Then: $\eta \overset{\text{def}}{:=} \max\limits_{m \neq j} \left| \dfrac{\lambda_j - \mu}{\lambda_m - \mu} \right|$ becomes smaller and smaller $\Rightarrow$ Converges faster and faster!

Definition: (Rayleigh quotient) Let $\vec{v} \neq \vec{0} \in \mathbb{R}^n$, $A \in M_{n\times n}$. Then, the Rayleigh quotient is defined as:

$$R(\vec{v}, A) = \frac{\vec{v}^* A \vec{v}}{\vec{v}^* \vec{v}}.$$

Remark: Let A be symmetric positive definite. Then: all eigenvalues:

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are real.

Then: $\lambda_n \leq R(\vec{v}, A) \leq \lambda_1$ and

$R(\vec{v}, A) = \lambda_1$ when $\vec{v} = \vec{v}_1$ = eigenvector of λ_1.

$R(\vec{v}, A) = \lambda_n$ when $\vec{v} = \vec{v}_n$ = eigenvector of λ_n

$R(\vec{v}, A)$ can be regarded as the approximation of eigenvalue λ_j, given that $\vec{v}$ is closed to $\vec{v}_j$.

Rayleigh Quotient Iteration

Let $A \in M_{n\times n}(\mathbb{C})$

Initiate $\vec{x}^{(0)}$ such that $\vec{x}^{(0)*}\vec{x}^{(0)} = 1$

Initiate μ_0 = initial guess of desired eigenvalue.

Solve: $(A - \mu_0 I)\vec{z}_1 = \vec{x}^{(0)}$

Let $\vec{x}^{(1)} = \dfrac{\vec{z}_1}{\|\vec{z}_1\|_2}$ $\left(\|\vec{x}\|_2 \overset{\text{def}}{=} \sqrt{\vec{x}^*\vec{x}} \right)$

Let $\mu_1 = R(\vec{x}^{(1)}, A) = \vec{x}^{(1)*} A \vec{x}^{(1)}$ (Improve μ_0 such that it is closer to an actual eigenvalue)

Keep iteration going!

Algorithm: (Rayleigh Quotient Iteration)

Input: $\vec{x}^{(0)}$ s.t. $\|\vec{x}^{(0)}\|_2 = 1$ and μ_0

Output: μ_k = eigenvalue

For $k = 0, 1, 2, \ldots$

Step 1: Solve $(A - \mu_k I)\vec{z}_{k+1} = \vec{x}^{(k)}$

Step 2: Let $\vec{x}^{(k+1)} = \dfrac{\vec{z}_{k+1}}{\|\vec{z}_{k+1}\|_2}$.

Step 3: $\mu_{k+1} = R(\vec{x}^{(k+1)}, A)$

Example: Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}$.

Eigenvalues: $\lambda_1 = 3+\sqrt{5}$, $\lambda_2 = 3-\sqrt{5}$, $\lambda_3 = -2$.

Want to find $3+\sqrt{5}$.

Let $\vec{x}^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mu_0 = 200$

Then: $\vec{x}^{(1)} \approx \begin{pmatrix} -0.57927 \\ -0.57348 \\ -0.57927 \end{pmatrix}$ with $\mu_1 = 5.3355$

$\vdots$

$\mu_3 = 5.2361 \approx 3+\sqrt{5}$!

Converges very fast!

Remark:

- RQI works for SPD A
- May or may not work for other A.