MATH 3310 Assignment 2
Due on February 22

1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2, & \text{for } x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$$

   Find the Fourier Transform of f.

2. Using Fourier Transform, find out *one* particular solution of the following partial differential equation:

$$\begin{aligned} u_{xx} - u_{tt} = 0 \quad & \text{on } \{(x,t) \mid t \geq 0\} \\ u(x,0) = e^{-|x|} & \end{aligned}$$

   Remark: You may express your final answer using an integral.

3. Recall the definitions of discrete and inverse discrete Fourier Transform from the lecture notes:
   Given: $f_0, f_1, \ldots, f_{n-1} \in \mathbb{C}$, the discrete Fourier transform is defined as

$$c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j e^{-i\frac{2jk\pi}{n}}$$

   for $k = 0, 1, 2, \ldots, n-1$. And the inverse discrete Fourier Transform:

$$f_j = \sum_{k=0}^{n-1} c_k e^{i\frac{2jk\pi}{n}}$$

   for $j = 0, 1, 2, \ldots, n-1$.
   Check that the inverse discrete Fourier Transform does recover the discrete Fourier Transform.

4. Let $f = \{f_i\}_{i=1}^n$ and $g = \{g_i\}_{i=1}^n$ be two sequences of points in $\mathbb{C}$ that are periodic. Define convolution by

$$(f * g)_i = \sum_{k=0}^{n-1} f_k g_{i-k}$$

Prove that for $k = 0, \dots, n-1$

$$\widehat{(f * g)}(k) = n\hat{f}(k)\hat{g}(k)$$

where $\hat{f} = \text{DFT}(f)$.

5. In addition to 1D DFT, we can also see an example that is 2D DFT. Consider this alternative definition for the DFT on $N \times N$ images:

$$\hat{f}(m,n) = DFT(f)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)e^{2\pi j\frac{mk+nl}{N}},$$

where $j = \sqrt{-1}$.

(a) Show that the inverse DFT (iDFT) is defined by

$$f(p,q) = iDFT(\hat{f})(p,q) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} \hat{f}(m,n)e^{-2\pi j\frac{pm+qn}{N}}.$$

(b) Determine the matrix U used to calculate the DFT of an $N \times N$ image, i.e. $\hat{f} = UfU$.

(c) Show that U is unitary (that is, $UU^* = U^*U = I$, where U^* is the conjugate transpose of U).

6. Consider the differential equation:

$$(**) \qquad a\frac{d^2u}{dx^2} + b\frac{du}{dx} = f(x) \text{ for } x \in (0, 2\pi),$$

where $a, b > 0$. Assume u and f are periodically extended to $\mathbb{R}$. Divide the interval $[0, 2\pi]$ into n equal portions and let $x_j = \frac{2\pi j}{n}$ for $j = 0, 1, 2, ..., n-1$.

Let $\mathbf{u} = (u(x_0), u(x_1), ..., u(x_{n-1}))^T$ and $\mathbf{f} = (f(x_0), f(x_1), ..., f(x_{n-1}))^T$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two $n \times n$ matrices, which are defined in such a way that:

$$(\mathcal{D}_1\mathbf{u})_j = \frac{u(x_{j+2}) - u(x_{j-2})}{4h} \quad \text{and} \quad (\mathcal{D}_2\mathbf{u})_j = \frac{u(x_{j+4}) - 2u(x_j) + u(x_{j-4})}{16h^2}.$$

for $j = 0, 1, 2, ..., n-1$.

(a) Explain why the differential equation (**) can be discretized as:

$$(***) \quad a\mathcal{D}_2\mathbf{u} + b\mathcal{D}_1\mathbf{u} = \mathbf{f}.$$

In other words, explain why $\mathcal{D}_1$ and $\mathcal{D}_2$ approximate $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ respectively.

(b) Prove that $\overrightarrow{e^{ikx}} := (e^{ikx_0}, e^{ikx_1}, ..., e^{ikx_{n-1}})^T$ is an eigenvector of both $\mathcal{D}_1$ and $\mathcal{D}_2$ for $k = 0, 1, 2, ..., n-1$. What are their corresponding eigenvalues? Please explain your answer with details.

(c) Show that $\{\overrightarrow{e^{ikx}}\}_{k=0}^{n-1}$ forms a basis for $\mathbb{C}^n$.

(d) Let $\mathbf{u} = \sum_{k=0}^{n-1} \hat{u}_k \overrightarrow{e^{ikx}}$ and $\mathbf{f} = \sum_{k=0}^{n-1} \hat{f}_k \overrightarrow{e^{ikx}}$, where $\hat{u}_k, \hat{f}_k \in \mathbb{C}$. If $\mathbf{u}$ satisfies (***), show that

$$(a\lambda_k^2 + b\lambda_k)\hat{u}_k = \hat{f}_k \text{ where } \lambda_k = i\frac{\sin(2kh)}{2h},$$

for $k = 0, 1, 2, ..., n-1$. Please explain your answer with details.