Week 2

MATH 2040B

September 22, 2020

1 Concepts

1. Linear combination and span (Class Note 3)

Note: In any vector space, $\operatorname{span}(\varnothing) = \{0\}$

2. Linear independence (Class Note 4)

Note: In any vector space, $\varnothing$ is linearly independent, while $\{0\}$ is linearly dependent(Why?).

2 Problems

1. In the vector space Mat $_{2\times 2}(\mathbb{C})$, determine whether the following statements are correct.

(a) The matrix $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$ is in the span of $\left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$

(b) The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is in the span of $\left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$

2. Determine whether the following sets are linearly independent.

(a) $\left\{ \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -5 \end{pmatrix} \right\}$ in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$

(b) $\{x^3 - 2x^2, -x^2 + 3x - 1, (x-1)^3\}$ in $\mathrm{P}_3(\mathbb{R})$

3. Let V be a real vector space, and W_1, W_2 are subspaces of V. The sum of W_1 and W_2 is defined as

$$W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$$

(a) Show that $W_1 + W_2$ is a subspace of V.

(b)If $W_1 = \operatorname{span}(S_1), W_2 = \operatorname{span}(S_2)$, show that $W_1 + W_2 = \operatorname{span}(S_1 \cup S_2)$

(c) Suppose that $W_1 \cap W_2 = \{0\}$. Then if $R_1 \subset W_1, R_2 \subset W_2$ are linearly independent subsets, show that $R_1 \cup R_2$ is also linearly independent.

4. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, and k be a positive integer such that $A^k \neq 0, A^{k+1} = 0$

(a) Show that $\left\{I, A, A^2, \dots, A^k\right\}$ is linearly independent.

(b) Show that $\left\{I, A+I, (A+I)^2, \dots, (A+I)^k\right\}$ is linearly independent.