Oct 22 : Lecture 15 :

Recall :

β : o.b. for V

$\dim(V) = n < \infty$

$$T \in \mathcal{L}(V)$$

$$T(\overset{\#}{v}{}^{\circ}) = \overset{\in F}{\lambda} v$$

e-vector e-value

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ \downarrow \phi_\beta & & \downarrow \phi_\beta \\ F^n & \xrightarrow{[T]_\beta} & F^n \end{array}$$

- e-values :

characteristic poly

$$f_T(t) = \det\big(\underset{\in M_{n\times n}(F)}{[T]_\beta - t I_n}\big)$$

$$= \det\big([T]_\gamma - t I_n\big)$$

$\hookrightarrow$ another o.b. for V.

zeros of $f_T(t)$ give all e-values of T

- e-vectors :

$v \in V$ is an e-vector associated with an e-value $\lambda \in F$ $\Leftrightarrow$ $\vec{v} \in N(T - \lambda I_V) \setminus \{\vec{0}\}$

$E_\lambda \overset{\text{def.}}{=} N(T - \lambda I_V)$

e-space of λ.

Examples :

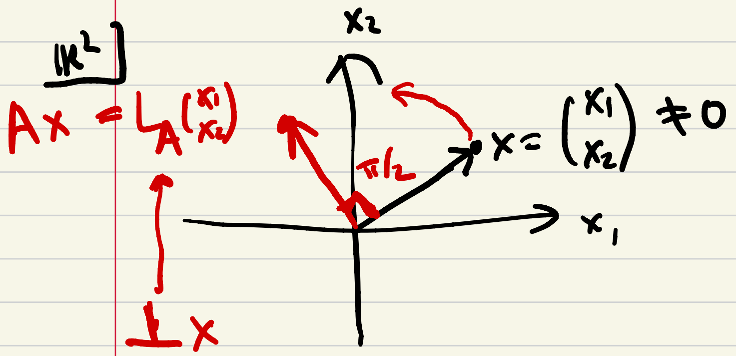
① $T_{\theta = \pi/2} = L_A$, $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\Bigg|_{\theta = \frac{\pi}{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Rotation by $\pi/2$

e-value & e-vector of $T = L_A \in \mathcal{L}(\mathbb{R}^2)$

$$\underline{L_A(x) = Ax} \overset{?}{=} \lambda x, \quad \lambda \in \mathbb{R}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

Geometric view:



$\Rightarrow$ no e-vectors & no e-values

Another view:

$$\beta \underset{\text{s.o.b.}}{=} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$[L_A]_\beta = A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$0 = f_T(t) = \det(A - tI_2) = \cdots = t^2 + 1$$

$\uparrow$ c.p. of L_A (or your A)

NO solution in $F = \mathbb{R}$

$\therefore$ no e-values

& no e-vectors.

However, choose $F = \mathbb{C}$

$$T \in \mathcal{L}(\mathbb{C}^2) : T = L_A, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$0 = f_T(t) = t^2 + 1, \quad t \in F = \mathbb{C}$$

$t = \pm i$ e-values of $T = L_A$

$$\lambda_1 = +i : \quad E_{\lambda_1} = N(L_A - \lambda_1 I_{\mathbb{C}^2})$$

$$= N(A - \lambda_1 I_2)$$

$$A - \lambda_1 I_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$$

$$\xrightarrow{\text{Row operators}} \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix}$$

$$E_{\lambda_1} = \left\{ \underbrace{a \begin{pmatrix} 1 \\ -i \end{pmatrix}}_{\substack{\downarrow\, a \neq 0 \\ \text{e-vetor of } L_A \text{ associated with } \lambda_1 = i}} : a \in \mathbb{C} \right\}$$

$\lambda_2 = -i$:

$$E_{\lambda_2} = N(A - \lambda_2 I_2)$$

$$= N\left(\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\right)$$

$$\cdots = \left\{ \underbrace{a \begin{pmatrix} 1 \\ i \end{pmatrix}}_{\substack{\downarrow\, a \neq 0 \\ \text{e-vector of } L_A \text{ associated with } \lambda_2 = -i.}} : a \in \mathbb{C} \right\}$$

$$\begin{array}{cc} \lambda_1 = i & \lambda_2 = -i \\ E_{\lambda_1} & E_{\lambda_2} \\ \downarrow & \downarrow \end{array}$$

$$\gamma \overset{\text{def.}}{=} \left\{ \begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix} \right\} \text{ is } \underbrace{\text{a basis}} \text{ for } \overset{\underbrace{\dim = 2}}{\mathbb{C}^2}$$

$(F = \mathbb{C})$

$$[T]_\gamma = [L_A]_\gamma = \begin{pmatrix} \lambda_1 = i & 0 \\ 0 & \lambda_2 = -i \end{pmatrix} \quad \text{diagonal}$$

Prop. $T \in \mathcal{L}(V)$

$\lambda_1, \dots, \lambda_k \in F$: distinct

$T(v_i) = \lambda_i v_i$, $\quad 0 \neq v_i \in V$, $\quad 1 \leq i \leq k$

e-vector $\in E_{\lambda_i}$

Then $\Longrightarrow$ $\{v_1, \dots, v_k\}$ is linearly independent.

Pf: Induction in $k = 1, 2, \dots$

$k=1$: $\{v_1\}$ is l. indep.

where $0 \neq v_1 \in V$ is e-vector of T

Assume "TRUE" for $k \geq 1$

to show "TRUE" for $k+1$.

i.e. to show $\{v_1, \dots, v_{k+1}\}$ is linearly independent.

where $T(v_i) = \lambda_i v_i$, $0 \neq v_i \in V$

$1 \leq i \leq k+1$

$\lambda_1, \dots, \lambda_{k+1}$: distinct.

Indeed,

let $0_V = \sum_{i=1}^{k+1} a_i v_i$, $\quad a_i \in F$,

Apply $T - \lambda_{k+1} I_V \in \mathcal{L}(V)$ to the above

$$0 = (T - \lambda_{k+1} I_V)(0_V)$$

$$= (T - \lambda_{k+1} I_V)\left(\sum_{i=1}^{k+1} a_i v_i\right)$$

$$= \sum_{i=1}^{k+1} a_i (T - \lambda_{k+1} I_V) v_i$$

$T(v_i) = \lambda_i v_i$

$$= \sum_{i=1}^{k+1} a_i \big(\underbrace{\lambda_i v_i - \lambda_{k+1} v_i}_{= (\lambda_i - \lambda_{k+1}) v_i} \big)$$

$\underset{\sim}{i = k+1}$

$$\therefore \quad 0_V = \sum_{i=1}^{k} a_i (\lambda_i - \lambda_{k+1}) v_i$$

Recall I.A. : $\{v_1, \cdots, v_k\}$ linearly indep.

$$\therefore \quad a_i \underbrace{(\lambda_i - \lambda_{k+1})}_{\neq 0} = 0, \quad 1 \le i \le k$$

$(\lambda_1, \cdots, \lambda_k, \lambda_{k+1}$: distinct $)$

$$\therefore \quad a_1 = \cdots = a_k = 0$$

Plug them back to $0 = \sum_{i=1}^{k+1} a_i v_i$,

$$a_{k+1} \underset{\neq 0}{v_{k+1}} = 0_V$$

$$\therefore \quad a_{k+1} = 0 \qquad \#.$$

Corollary : $T \in \mathcal{L}(V)$, $\dim(V) = n < \infty$

T has n distinct e-values

$\Rightarrow$ T is diagonalizable

i.e. ....

pf. Let $\lambda_1, \cdots, \lambda_n$: distinct e-values

$v_1, \cdots, v_n$: e-vectors $(\neq 0)$, resp.

By Prop., $\beta \stackrel{\text{def.}}{=} \{v_1, \cdots, v_n\}$: a basis for V.

$\therefore$ T is diagonalizable.

Def. $T \in \mathcal{L}(V)$, $\dim(V) < \infty$.

λ : e-value of T

$f_T(t) = \det(T - \lambda I_V)$: c.p.

$$\left.\begin{array}{l}\text{algebraic multiplicity} \\ \text{of } \lambda \\ = \mu_T(\lambda)\end{array}\right\} \stackrel{\text{def.}}{=} \max\{k \geq 1 : (t-\lambda)^k \mid f_T(t)\}$$

e.g. $f_T(t) = \underbrace{(t-1)^{3}(t-4)^{4}(t-5)^{7}}$

A.M. of $\lambda = 1$ is 3

$\lambda = 4$ is 4

$\lambda = 5$ is 7. #