Lecture 3:

Recall: • Linear combination of S :

$$\vec{v} = \underset{\substack{\wedge \\ F}}{a_1} \underset{\substack{\wedge \\ S}}{\vec{v}_1} + \underset{\substack{\wedge \\ F}}{a_2} \underset{\substack{\wedge \\ S}}{\vec{v}_2} + \dots + \underset{\substack{\wedge \\ F}}{a_n} \underset{\substack{\wedge \\ S}}{\vec{v}_n}$$

• a_i's are called coefficients

• $\operatorname{Span}(S) = \{ a_1\vec{v}_1 + \dots + a_n\vec{v}_n : a_i \in F,\ i = 1, 2, \dots, n,\ n \in \mathbb{N},\ \vec{v}_j \in S \}$

Theorem: Let $S \subset V$ be a subset of a vector space V over F.
Then, span(S) is the smallest subspace of V consisting S.

(If W is a subspace containing S, then $\text{Span}(S) \subset W$)

Proof: If $S = \phi$, then $\text{span}(S) = \{\vec{0}\}$. $\leftarrow$ subspace. is contained in any other subspace.

The result holds.

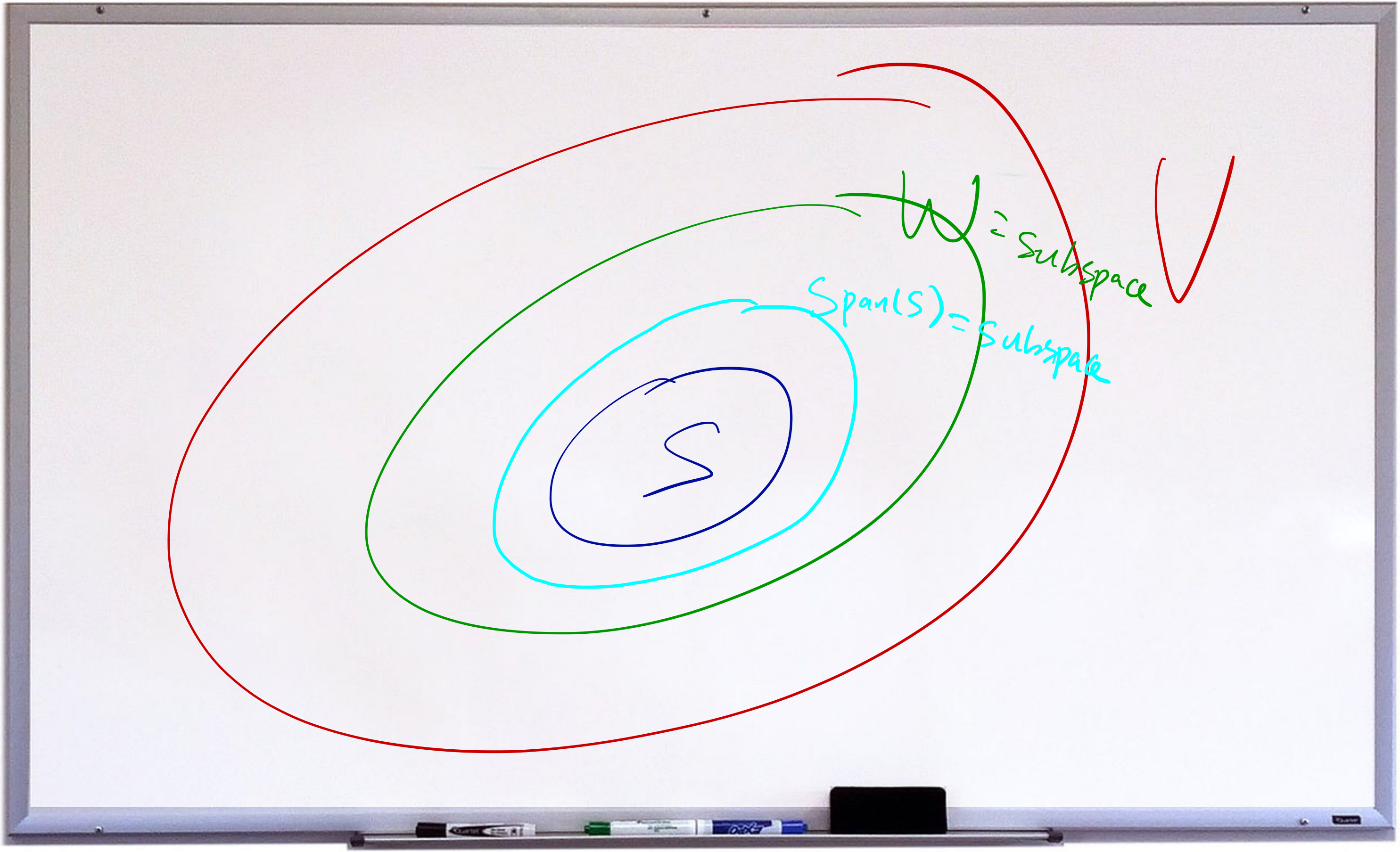
Suppose $S \neq \phi$. Let $\vec{z} \in S$. Then $\vec{0}_V = \underset{F}{0}\,\vec{z} \in \text{Span}(S)$

If $\vec{x}, \vec{y} \in \text{Span}(S)$, then we can write:

$$\vec{x} = a_1\vec{u}_1 + \dots + a_m\vec{u}_m, \quad \vec{u}_1, \vec{u}_2, \dots, \vec{u}_m \in S$$
$$a_1, \dots, a_m \in F$$

$$\vec{y} = b_1\vec{v}_1 + \dots + b_n\vec{v}_n, \quad \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in S$$
$$b_1, \dots, b_n \in F$$

$$\vec{x} + \vec{y} = a_1\vec{u}_1 + \dots + a_m\vec{u}_m + b_1\vec{v}_1 + \dots + b_n\vec{v}_n \in \text{Span}(S)$$



W = Subspace
V
Span(S) = Subspace
S

$$c\vec{x} = \underset{\overset{\Uparrow}{F}}{c}\,\vec{x} = \underset{\overset{\Uparrow}{F}}{(ca_1)}\vec{u}_1 + \cdots + \underset{\overset{\Uparrow}{F}}{(ca_m)}\vec{u}_m \in \operatorname{Span}(S)$$

$\therefore$ Span(S) is a subspace.

Now, let $W \subset V$ be a subspace of V containing S.

WANT TO SHOW: $\operatorname{Span}(S) \subset W$.

Let $\vec{x} \in \operatorname{Span}(S)$. Write $\vec{x} = \underset{\overset{\Uparrow}{F}}{a_1}\,\underset{\overset{\Uparrow}{\underset{\overset{\cap}{W}}{S}}}{\vec{u}_1} + \cdots + \underset{\overset{\Uparrow}{F}}{a_m}\,\underset{\overset{\Uparrow}{\underset{\overset{\cap}{W}}{S}}}{\vec{u}_m}$

$\because$ $S \subset W$, $\therefore$ $\vec{u}_1 \in W$, $\vec{u}_2 \in W$, ..., $\vec{u}_m \in W$

$\therefore$ $\vec{x} = a_1\vec{u}_1 + \cdots + a_m\vec{u}_m \in W$ (Why??]
($\because$ W is a subspace)

$\therefore$ $\operatorname{Span}(S) \subset W$.

QED.

Definition: We say a subset $S \subset V$ of a vector space V over F **spans (or generates)** V if $V = \mathrm{span}(S)$.

In this case, S is called a **spanning set (or generating set)** for V.

e.g.

- $\{\vec{e}_1, \ldots, \vec{e}_n\}$ spans F^n
- $\{1, x, \ldots, x^n, \ldots\}$ spans $P(F)$

Linear independence

Definition: Let V be a vector space over F. A subset $S \subset V$ is said to be linearly dependent if $\exists$ distinct $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \in S$ and scalars $a_1, a_2, \ldots, a_n \in F$, not all zero, s.t.

$$a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n = \vec{0}$$

Otherwise, it is said to be linearly independent.

e.g.

- The empty set $\phi \subset V$ is linearly independent.
- If $\vec{0} \in S$, the S is linearly dependent $\left(5\underset{\in S}{\vec{0}} = \vec{0} \right)$
- If $S = \{\vec{u}\}$ and $\vec{u} \neq \vec{0}$, then S is linearly independent.

$$\left(\begin{array}{l} \lambda \overset{\neq \vec{0}}{\vec{u}} = \vec{0} \\ \Rightarrow \lambda = 0 \end{array} \right)$$

Proposition: Let $S \subset V$ be a subset of a vector space V. Then, the following are equivalent.

(1) S is linearly independent

(2) Each $\vec{x} \in \operatorname{span}(S)$ can be expressed in a unique way as a linear combination of vectors of S.

(3) The only representations of $\vec{0}$ as linear combinations of vectors of S are trivial representations, i.e., if

$$\vec{0} = a_1 \vec{u}_1 + \dots + a_n \vec{u}_n \quad \text{for}$$

some $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \in S$, $a_1, a_2, \dots, a_n \in F$, then we must have $a_1 = a_2 = \dots = a_n = 0$