Lecture 2: Subspaces

Recap:

Definition: A vector space over F is a set V equipped w/ two operations:

(addition) $+ : V \times V \to V$, $(\underset{V}{\vec{x}}, \underset{V}{\vec{y}}) \mapsto \vec{x} + \vec{y} \in V$

(Scalar multiplication) $\cdot : F \times V \to V$, $(\overset{}{\underset{F}{a}}, \underset{V}{\vec{x}}) \mapsto a\vec{x} \in V$

satisfying 8 properties:

$+$

$$(VS1):\quad \vec{x}+\vec{y}=\vec{y}+\vec{x}\qquad \forall \vec{x},\vec{y}\in V$$

$$(VS2):\quad (\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z})\qquad \forall \vec{x},\vec{y},\vec{z}\in V$$

$$(VS3):\quad \exists\ \vec{0}\in V \text{ s.t. } \vec{x}+\vec{0}=\vec{x}\qquad \forall \vec{x}\in V$$

$$(VS4):\quad \forall \vec{x}\in V,\ \exists\ \vec{y}\in V \text{ s.t. } \vec{x}+\vec{y}=\vec{0}\qquad \text{(inverse)}$$

$\cdot$

$$(VS5):\quad \underset{\substack{\uparrow\\ F}}{1}\,\vec{x}=\vec{x}\qquad \forall \vec{x}\in V$$

$$(VS6):\quad (\underset{\substack{\uparrow\\ F}}{a}\,\underset{\substack{\uparrow\\ F}}{b})\vec{x}=a(b\vec{x})\qquad \forall a,b\in F,\ \forall \vec{x}\in V$$

$+/\cdot$

$$(VS7):\quad \underset{\substack{\uparrow\\ F}}{a}\,(\underset{\substack{\uparrow\\ V}}{\vec{x}}+\underset{\substack{\uparrow\\ V}}{\vec{y}})=a\vec{x}+a\vec{y}\qquad \forall a\in F,\ \forall \vec{x},\vec{y}\in V$$

$$(VS8):\quad (a+b)\vec{x}=a\vec{x}+b\vec{x}\qquad \forall a,b\in F,\ \forall \vec{x}\in V$$

<u>Remark</u>: an element in F is called <u>scalar</u>.
" " " V is called <u>vector</u>.

Proposition: Let V be a vector space over F. Then:

(a) The element $\vec{0}$ in (VS 3) is unique, called <u>zero vector</u>.

(b) $\forall \vec{x} \in V$, the element $\vec{y}$ in (VS4) is unique, called the additive inverse of $\vec{x}$ (Denote as $-\vec{x}$)

(c) $\vec{x} + \vec{z} = \vec{y} + \vec{z} \Rightarrow \vec{x} = \vec{y}$ (Cancellation law)

(d) $\underset{\substack{\uparrow \\ F}}{0}\,\vec{x} = \vec{0} \quad \forall \vec{x} \in V.$

(e) $(-a)\vec{x} = -(a\vec{x}) = a(-\vec{x}) \quad \forall a \in F, \forall \vec{x} \in V$

(f) $\underset{\substack{\uparrow \\ F}}{a}\,\vec{0} = \vec{0} \quad \forall a \in F.$

Proof: (a). If $\vec{0}$ and $\vec{0}\,'$ are two elements satisfying (VS 3).
$\vec{0} \in V$, $\vec{0}\,' \in V$

Then:
$$\left.\begin{aligned}\vec{0} &= \vec{0} + \vec{0}\,' \\ \vec{0}\,' &= \vec{0}\,' + \vec{0} \underset{(VS\,1)}{=} \vec{0} + \vec{0}\,'\end{aligned}\right\} \Rightarrow \vec{0} = \vec{0}\,'$$

(b). Given $\vec{x} \in V$. Suppose we have $\vec{y}, \vec{y}\,' \in V$ satisfying (VS 4). Then:
$$\vec{x} + \vec{y} = \vec{0} = \vec{x} + \vec{y}\,'$$

Then:
$$\vec{y} = \vec{y} + \vec{0} = \vec{y} + (\vec{x} + \vec{y}\,') \overset{(VS\,2)}{=} (\underbrace{\vec{y} + \vec{x}}_{\vec{0}}) + \vec{y}\,' = \vec{y}\,'$$

(c) $\vec{x} + \vec{z} = \vec{y} + \vec{z}$

$$\Rightarrow (\vec{x} + \vec{z}) + (-\vec{z}) = (\vec{y} + \vec{z}) + (-\vec{z})$$

$$\Rightarrow \vec{x} + \underbrace{(\vec{z} + (-\vec{z}))}_{\vec{0}} = \vec{y} + \underbrace{(\vec{z} + (-\vec{z}))}_{\vec{0}}$$

$$\Rightarrow \vec{x} = \vec{y}$$

(d) $$0\vec{x} = (0+0)\vec{x} \overset{(V8)}{=} 0\vec{x} + 0\vec{x} \overset{\text{by (c)}}{\Rightarrow} 0\vec{x} = \vec{0}$$

$$0\vec{x} = 0\vec{x} + \vec{0}$$

(e) $$\vec{0} \underset{(\text{by } d)}{=} \underbrace{(a-a)}_{=\,(a+(-a))}\vec{x} = a\vec{x} + (-a)\vec{x} \Rightarrow (-a)\vec{x} = -(a\vec{x})$$

Other part: leave as exercise.

(f)

$$
\begin{array}{l}
a\vec{0} = a(\vec{0}+\vec{0}) \overset{(VS\,7)}{=} a\vec{0} + a\vec{0} \\
\parallel \\
a\vec{0} + \vec{0}
\end{array}
$$

$$\underset{(\text{by } (c))}{\Longrightarrow} \quad a\vec{0} = \vec{0}$$

Subspace

Definition: A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F under the same addition and scalar multiplication inherited from V.

Proposition: Let V be a vector space over F. A subset $W \subset V$ is a subspace iff the following 3 conditions holds:

(a) $\vec{0}_V \in W$

(b) $\vec{x} + \vec{y} \in W$, $\forall \vec{x}, \vec{y} \in W$ (closed under addition)

(c) $a\vec{x} \in W$, $\forall a \in F, \vec{x} \in W$ (closed under scalar multiplication)

Proof: ($\Rightarrow$) If $W \subset V$ is a subspace, then (b) and (c) must hold because W is a vector space.

W has an zero element ($\because$ W is a vector space)
$\vec{0}_W$

Then: $\vec{0}_W + \vec{0}_W = \vec{0}_W$ in V.

$$\vec{0}_W + \vec{0}_W = \vec{0}_W = \vec{0}_W + \vec{0}_V$$

(Cancellation law) $\Rightarrow$ $\underset{\substack{\uparrow \\ W}}{\vec{0}_W} = \vec{0}_V$

($\Leftarrow$) If (a)-(c) hold, then addition and scalar multiplication are well-defined on W (by (b) and (c)) and (VS3) follows from (a).

(VS1), (VS2), (VS5)-(VS8) hold for V, so they hold for W as well.

Remain to check (VS4).

Let $\vec{x} \in W^{\subset V}$. Then, we have $-\vec{x} \in V$.

But $-\vec{x} = (\overset{\in F}{-1})\overset{\in W}{\vec{x}} \in W$

(by (c))

$\therefore$ W is a vector space over F under the same addition and scalar multiplication.

Examples:

- For any vector space V,

  $\underset{W}{\{\vec{0}\}} \subset V$; $V \subset V$ (trivial subspaces)

- For $V = M_{n\times n}(F)$,

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in W_1 = \{\text{diagonal matrices}\} \subset V$

$+ \quad W_2 = \{A \in M_{n\times n}(F) : \det(A) = 0\} \subset V$ NOT subspace

$(\det(A+B) \neq \det(A) + \det(B))$

$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \in W_3 = \{A \in M_{n\times n}(F) : \operatorname{tr}(A) = 0\} \subset V$

$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$

$\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$

- For $V = P(F)$ $(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n)$

$$P_n(F) \overset{\text{def}}{:=} \{ f \in P(F) : \deg(f) \le n \}$$ is a subspace

$$W \overset{\text{def}}{=} \{ f \in P(F) : \deg(f) = n \}$$

- Consider $V = F^n = \{(x_1, x_2, \dots, x_n) : x_j \in F \text{ for } j = 1, 2, \dots, n\}$

Consider linear system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases} \iff A\vec{x} = \vec{b}$$

gives a subset, the solution set $S \subset V$.

Is S a subspace??

NO if $(b_1, b_2, \dots, b_m) \neq \vec{0}$

Yes iff $\vec{b} = \vec{0}$. (Null space)

Theorem: Any intersection of subspaces of a vector space V is a subspace of V.

Proof: Let $\{W_i\}_{i \in I}$ be a collection of subspaces of V.

Set $W \overset{\text{def}}{:=} \bigcap_{i \in I} W_i \subset V$.

$\because \vec{0}_V \in W_i$ for $\forall i \in I$ $\therefore \vec{0}_V \in W$.

For any $\vec{x} \in W$ and $\vec{y} \in W$, we have $\vec{x} \in W_i$, $\vec{y} \in W_i$ for $\forall i$.

Then: $\vec{x} + \vec{y} \in W_i$ for $\forall i \in I \Rightarrow \vec{x} + \vec{y} \in W$

For $\vec{x} \in W$, $a \in F$, we have $a\vec{x} \in W_i$ for all $i \in I$.

$\therefore a\vec{x} \in W$.

$\therefore W$ is a subspace.

Question: W_1 = subspace ; W_2 = subspace

$\Downarrow$

$W_1 \cap W_2$ is subspace

Is $W_1 \cup W_2$ a subspace?? No in general!

Linear combination and Span

Definition: Let V be a vector space over F and $S \subset V$ a non-empty subset.

- We say a vector $\vec{v} \in V$ is a linear combination of vectors of S if $\exists$ $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \in S$ and $a_1, a_2, \dots, a_n \in F$ s.t.

$$\vec{v} = a_1 \vec{u}_1 + a_2 \vec{u}_2 + \dots + a_n \vec{u}_n.$$

Remark: $\vec{v}$ is usually called a linear combination of $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ and $a_1, a_2, \dots, a_n$ the coefficients of the linear combination.

- The span of S, denoted as $\mathrm{Span}(S)$, is the set of all linear combination of vectors of S:

$$\mathrm{Span}(S) \overset{def}{=} \{ a_1 \vec{u}_1 + a_2 \vec{u}_2 + \dots + a_n \vec{u}_n : a_j \in F, \vec{u}_j \in S \text{ for } j = 1, 2, \dots, n, n \in \mathbb{N} \}$$

Remark: By convention, $\text{span}(\underset{\substack{\| \\ \text{empty set}}}{\phi}) \overset{\text{def}}{=} \{\vec{0}\}$

- $1 \in \text{Span}\{1+x^2, \; 1-x^2\}$

$\cancel{\alpha}$

x

Example:

- $F^n = \text{span}\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ where $\vec{e}_j = (0, 0, \dots, \underset{\substack{\uparrow \\ j\text{-th}}}{1}, 0 \dots 0)$

- $P(F) = \text{span}\{1, x, x^2, \dots, \cancel{x^n}, \dots\}$

- $M_{n\times n}(F) = \text{span}(S)$,

$$S = \left\{ E_{ij} = \begin{array}{cc} & \scriptstyle \downarrow j \\ & \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} \end{array} \leftarrow i \;:\; 1 \le i, j \le n \right\}$$

- Given $\vec{u}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}$, $\dots$, $\vec{u}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}$

$$\underbrace{x_1\vec{u}_1 + x_2\vec{u}_2 + \dots + x_n\vec{u}_n}_{= \vec{v}}$$

$$\Updownarrow$$

Then: $\underset{\substack{\shortparallel \\ (v_1, v_2, \dots, v_n)}}{\vec{v}} \in \text{Span}(\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\})$ iff:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = v_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = v_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = v_n \end{cases}$$

is consistent. (has sol)