Lecture 17: **Recap:**

Theorem: Let $T: V \to V$ be a linear operator on a finite-dim vector space V and let $W \subset V$ be T-cyclic subspace of V generated by $\vec{v} \neq \vec{0} \in V$. $(W = \text{span}\{\vec{v}, T(\vec{v}), T^2(\vec{v}), \dots\})$

Let $k = \dim(W)$. Then:

(a) $\{\vec{v}, T(\vec{v}), T^2(\vec{v}), \dots, T^{k-1}(\vec{v})\}$ is a basis for W

(b) If $a_0\vec{v} + a_1 T(\vec{v}) + a_2 T^2(\vec{v}) + \dots + a_{k-1}T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0}$,

then the characteristic polynomial of $T|_W$ is:

$$f_{T_W}(t) = (-1)^k(a_0 + a_1 t + a_2 t^2 + \dots + a_{k-1}t^{k-1})$$

Part (a) has been proven last time!!

(b) By (a), $\beta = \{\vec{v}, T(\vec{v}), \ldots, T^{k-1}(\vec{v})\}$ is an ordered basis for W.

Let $a_0, \ldots, a_{k-1} \in F$ s.t.

$$a_0\vec{v} + a_1 T(\vec{v}) + \cdots + a_{k-1} T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0}$$

$$\Rightarrow T^k(\vec{v}) = -a_0\vec{v} - a_1 T(\vec{v}) - \cdots - a_{k-1} T^{k-1}(\vec{v}).$$

Then: $$[T]_\beta = \begin{pmatrix} | & | & & | \\ [T(\vec{v})]_\beta & [T(T(\vec{v}))]_\beta & \cdots & [T^k(\vec{v})]_\beta \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & & 0 & -a_0 \\ 1 & 0 & & \vdots & -a_1 \\ 0 & 1 & & \vdots & \vdots \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & & 1 & -a_{k-1} \end{pmatrix}$$

$$f_{T_W}(t) = \det\left(\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & \ddots & 0 & \cdots 0 & -a_1 \\ & \ddots & \ddots & & \vdots \\ & & & 1 \quad 0 & -a_{k-1} \end{pmatrix} - t I_k \right)$$

$$= (-1)^k \left(a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k \right) \quad \text{(HW)}$$

Theorem: (Cayley - Hamilton) Let T be a linear operator on a finite-dim. vector space V and let $f(t) = f_T(t)$ be a char poly of T. Then: $f(T) =$ zero transformation.

(Char poly "kills" the linear operator T)

Remark: $f(t) = a_0 1 + a_1 t + a_2 t^2 + \dots + a_n t^n$

$$f(T) = a_0 I + a_1 T + a_2 T^2 + \dots + a_n T^n$$

Proof: We want to show $f(T)(\vec{v}) = \vec{0}$ for all $\vec{v} \in V$.

$f(T)(\vec{0}) = \vec{0}$ ($\because$ $f(T)$ is linear)

So, suppose $\vec{v} \neq \vec{0}$. Let W = T-cyclic subspace generated by $\vec{v}$.

Let $k = \dim(W)$

By Thm we have shown last time:

$\exists\ a_0, a_1, \ldots, a_{k-1} \in F$ such that:

$$\begin{cases} \cdot\ a_0\vec{v} + a_1 T(\vec{v}) + \ldots + a_{k-1}T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0} \\ \cdot\ g(t) \stackrel{\text{def}}{=} f_{T|_W} = (-1)^k\left(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k\right) \end{cases}$$

$$\begin{cases} \cdot\ a_0 \vec{v} + a_1 T(\vec{v}) + \dots + a_{k-1} T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0} \\ \cdot\ g(t) \overset{\text{def}}{=} f_{T|_W} = (-1)^k (a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + t^k) \end{cases}$$

$$\Downarrow$$

$$g(T)(\vec{v}) = \vec{0}$$

Now, $g(t) \mid f(t)$ $\overset{\text{implies}}{\rightsquigarrow}$ $\exists\, q(t)$ s.t. $f(t) = q(t)\, g(t)$

$$\Downarrow$$

$$f(T) = q(T)\, g(T)$$

$$\|$$

$$q(T) \circ g(T)$$

$$\therefore\ f(T)(\vec{v}) = q(T) \circ \underbrace{g(T)(\vec{v})}_{\vec{0}} = \vec{0}$$

$$\overset{\in V}{q(T)(\vec{v})} \ \circledast\ \overset{\in V}{g(T)(\vec{v})} = 0\ ?$$

Corollary: Let $A \in M_{n\times n}(F)$ and $f(t)$ be its char. poly. Then: $f(A) = O$, the zero matrix.

Inner product and norm

Assume $F = \mathbb{R}$ or $\mathbb{C}$.

Definition: Let V be a vector space over F. An inner product on V is a map $\langle \cdot , \cdot \rangle : V \times V \to F$ s.t. $\forall \vec{x}, \vec{y}, \vec{z} \in V$ and $c \in F$, it satisfies:

(a) $\langle \vec{x} + \vec{z}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$

(b) $\langle c\vec{x}, \vec{y} \rangle = c \langle \vec{x}, \vec{y} \rangle$

(c) $\overline{\langle \vec{x}, \vec{y} \rangle} = \langle \vec{y}, \vec{x} \rangle$

(d) $\underbrace{\langle \vec{x}, \vec{x} \rangle}_{\in \mathbb{R}} > 0$ if $\vec{x} \neq \vec{0}$

Remark:

- (a), (b) say that the inner product is linear in its argument.
- If $F = \mathbb{R}$, $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

Example:

- For $\vec{x} = (a_1, a_2, \dots, a_n)$, $\vec{y} = (b_1, b_2, \dots, b_n) \in F^n$
  $(F = \mathbb{R}, \mathbb{C})$

We have: standard inner product

$$\langle \vec{x}, \vec{y} \rangle :\overset{def}{=} \sum_{i=1}^{n} a_i \overline{b_i}$$

- If $\langle \cdot, \cdot \rangle$ is an inner product on V, and $r > 0$,

then: $\langle \vec{x}, \vec{y} \rangle' :\overset{def}{=} r \langle \vec{x}, \vec{y} \rangle$ is another inner product on V.

- Let $V = C([0,1])$ be vector space of real-valued continuous functions on $[0,1]$. Then: for $f, g \in V$,

$$\langle f, g \rangle \overset{\text{def}}{=} \int_0^1 f(t) g(t)\, dt$$ defines an inner product on V.

$(F = \mathbb{R}, \mathbb{C})$

- Let $V = M_{n \times n}(F)$. For $A, B \in V$, we define:

sum of diagonal entries.

$$\langle A, B \rangle \overset{\text{def}}{=} \operatorname{tr}(B^* A)$$

where B^* is the conjugate transpose of B defined by:

$$B^* = \overline{B}^T$$

For $A, B, C \in V$ and $\lambda \in F$, we check:

(a)
$$\langle A+B, C\rangle = \operatorname{tr}\big(C^*(A+B)\big) = \operatorname{tr}(C^*A + C^*B)$$
$$= \operatorname{tr}(C^*A) + \operatorname{tr}(C^*B)$$
$$= \langle A, C\rangle + \langle B, C\rangle$$

(b)
$$\langle \lambda A, B\rangle = \operatorname{tr}\big(B^*(\lambda A)\big) = \operatorname{tr}\big(\lambda(B^*A)\big)$$
$$= \lambda \operatorname{tr}(B^*A)$$
$$= \lambda\langle A, B\rangle$$

(c)
$$\overline{\langle A, B\rangle} = \overline{\operatorname{tr}(B^*A)} = \operatorname{tr}\big(\overline{\underbrace{B^*}_{=\overline{B^T}}A}\big) = \operatorname{tr}(B^T\bar{A})$$

$$\operatorname{Tr}(C) = \operatorname{Tr}(C^T)$$

$$= \operatorname{tr}\big((B^T\bar{A})^T\big) = \operatorname{tr}\big(\bar{A}^T\underbrace{(B^T)^T}_{B}\big)$$
$$= \operatorname{tr}(A^*B) = \langle B, A\rangle.$$

(d) $\langle A, A \rangle = tr(A^*A) = \sum_{i=1}^{n} (A^*A)_{ii}$

$$= \sum_{i=1}^{n} \Big(\sum_{k=1}^{n} \underset{\overline{A}^T}{\underset{\shortparallel}{(A^*)_{ik}}} \, A_{ki} \Big)$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} \underbrace{\overline{A}_{ki} A_{ki}}_{\underset{|A_{ki}|^2}{\shortparallel}}$$

$$\langle A, A \rangle = \sum_{i=1}^{n} \sum_{k=1}^{n} |A_{ki}|^2 \geq 0$$

and $\langle A, A \rangle = 0$ iff $A_{ki} = 0 \ \forall \, k, i$ (i.e. $A = 0$)

Definition: A vector space V equipped with an inner product is called an inner product space.

If $F = \mathbb{C}$, we call V a complex inner product space.

If $F = \mathbb{R}$, we call V a real inner product space.

Proposition: Let V be an inner product space. Then, $\forall \vec{x}, \vec{y}, \vec{z} \in V$ and $\forall c \in F$, we have:

(a) $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

(b) $\langle \vec{x}, c\vec{y} \rangle = \overline{c} \langle \vec{x}, \vec{y} \rangle$

(c) $\langle \vec{x}, \vec{0} \rangle = \langle \vec{0}, \vec{x} \rangle = 0$

(d) $\langle \vec{x}, \vec{x} \rangle = 0$ iff $\vec{x} = \vec{0}$

(e) If $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle$ for $\forall \vec{x} \in V$, then $\vec{y} = \vec{z}$.

<u>Proposition:</u> Let V be an inner product space. Then, $\forall \vec{x}, \vec{y}, \vec{z} \in V$ and $\forall c \in F$, we have:

(a) $\langle \vec{x}, \vec{y}+\vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

(b) $\langle \vec{x}, c\vec{y} \rangle = \bar{c} \langle \vec{x}, \vec{y} \rangle$

(c) $\langle \vec{x}, \vec{0} \rangle = \langle \vec{0}, \vec{x} \rangle = 0$

(d) $\langle \vec{x}, \vec{x} \rangle = 0$ iff $\vec{x} = \vec{0}$

(e) If $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle$ for $\forall \vec{x} \in V$, then $\vec{y} = \vec{z}$.

If A, then B $\iff$ If not B, then not A.

Proof:

(a) $\langle \vec{x}, \vec{y}+\vec{z} \rangle = \overline{\langle \vec{y}+\vec{z}, \vec{x} \rangle}$

$= \overline{\langle \vec{y}, \vec{x} \rangle + \langle \vec{z}, \vec{x} \rangle}$

$= \overline{\langle \vec{y}, \vec{x} \rangle} + \overline{\langle \vec{z}, \vec{x} \rangle} = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

(b) $\langle \vec{x}, c\vec{y} \rangle = \overline{\langle c\vec{y}, \vec{x} \rangle} = \overline{c\langle \vec{y}, \vec{x} \rangle} = \bar{c}\, \overline{\langle \vec{y}, \vec{x} \rangle}$

$= \bar{c}\, \langle \vec{x}, \vec{y} \rangle$

(c) $\langle \vec{x}, \vec{0} \rangle = \langle \vec{x}, \vec{0}+\vec{0} \rangle = \langle \vec{x}, \vec{0} \rangle + \langle \vec{x}, \vec{0} \rangle$

So, $\langle \vec{x}, \vec{0} \rangle = 0$. Similarly, $\langle \vec{0}, \vec{x} \rangle = 0$

(d) If $\vec{x} = \vec{0}$, then $\langle \vec{x}, \vec{x} \rangle = 0$ by (c)

If $\vec{x} \neq \vec{0}$, then $\langle \vec{x}, \vec{x} \rangle > 0$ by definition.

(e) If $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle$ for all $\vec{x} \in V$.

then $\langle \vec{x}, \vec{y} - \vec{z} \rangle = 0 \quad \forall \vec{x} \in V$.

In particular, we can choose $\vec{x} = \vec{y} - \vec{z}$.

Then: $\langle \vec{y} - \vec{z}, \vec{y} - \vec{z} \rangle = 0 \Rightarrow \vec{y} - \vec{z} = \vec{0}$ (by (d))

$$\Rightarrow \vec{y} = \vec{z}.$$

<u>Remark:</u> (a) + (b) together say that the inner product is <u>conjugate linear</u> in the second argument.

<u>Definition</u>: Let V be an inner product space. For $\vec{x} \in V$, we can define the <u>length</u> or <u>norm</u> of $\vec{x}$ by:

$$\|\vec{x}\| \overset{\text{def}}{:=} \sqrt{\langle \vec{x}, \vec{x} \rangle}$$

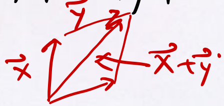
<u>Proposition</u>: Let V be an inner product space over F. Then, $\forall \vec{x}, \vec{y} \in V$ and $\forall c \in F$, we have:

(a) $\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$

(b) $\|\vec{x}\| \geq 0$, and $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$.

(c) $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \, \|\vec{y}\|$ (Cauchy-Schwarz Inequality)

(d) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ (Triangle inequality)



Proof:

(a) $\|c\vec{x}\| = \sqrt{\langle c\vec{x}, c\vec{x}\rangle} = \sqrt{\underset{\overset{\shortparallel}{|c|^2}}{c\,\bar{c}}\,\langle \vec{x}, \vec{x}\rangle}$

$= |c|\sqrt{\langle \vec{x}, \vec{x}\rangle} = |c|\,\|\vec{x}\|.$

(b) $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x}\rangle} \geq 0$ (by definition)

$\|\vec{x}\| = 0 \iff \langle \vec{x}, \vec{x}\rangle = 0$ iff $\vec{x} = \vec{0}$

(c) If $\vec{y} = \vec{0}$, then the inequality holds.

So, assume $\vec{y} \neq \vec{0}$. Now, $\forall c \in F$,

$0 \leq \|\vec{x} - c\vec{y}\|^2 = \langle \vec{x} - c\vec{y}, \vec{x} - c\vec{y}\rangle = \langle \vec{x}, \vec{x} - c\vec{y}\rangle - c\langle \vec{y}, \vec{x} - c\vec{y}\rangle$

$0 \leq \langle \vec{x}, \vec{x}\rangle - \bar{c}\langle \vec{x}, \vec{y}\rangle - c\langle \vec{y}, \vec{x}\rangle + c\,\bar{c}\,\langle \vec{y}, \vec{y}\rangle$

$$0 \le \langle \vec{x}, \vec{x} \rangle - \bar{c} \langle \vec{x}, \vec{y} \rangle - c \langle \vec{y}, \vec{x} \rangle + c\bar{c} \langle \vec{y}, \vec{y} \rangle$$

Set $c = \dfrac{\langle \vec{x}, \vec{y} \rangle}{\langle \vec{y}, \vec{y} \rangle}$.

Then: $0 \le \langle \vec{x}, \vec{x} \rangle - \dfrac{|\langle \vec{x}, \vec{y} \rangle|^2}{\langle \vec{y}, \vec{y} \rangle} = \|\vec{x}\|^2 - \dfrac{|\langle \vec{x}, \vec{y} \rangle|^2}{\|\vec{y}\|^2}$

$$\Longleftrightarrow \quad |\langle \vec{x}, \vec{y} \rangle|^2 \le \|\vec{x}\|^2 \|\vec{y}\|^2$$

$$|\langle \vec{x}, \vec{y} \rangle| \le \|\vec{x}\| \|\vec{y}\|.$$

(d) $\|\vec{x}+\vec{y}\|^2 = \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle$

$\langle \vec{y}, \vec{x} \rangle = \overline{\langle \vec{x}, \vec{y} \rangle}$

$$= \|\vec{x}\|^2 + 2\,\mathrm{Re}\langle \vec{x}, \vec{y} \rangle + \|\vec{y}\|^2$$

$$\leq \|\vec{x}\|^2 + 2\,|\langle \vec{x}, \vec{y} \rangle| + \|\vec{y}\|^2$$

$$\leq \|\vec{x}\|^2 + 2\,\|\vec{x}\|\,\|\vec{y}\| + \|\vec{y}\|^2$$

$$= (\|\vec{x}\| + \|\vec{y}\|)^2$$

$\left(a \leq \sqrt{a^2+b^2} \quad \boxed{z = a+ib} \right)$

$$\Rightarrow \quad \|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|.$$