Lecture 12:

Def: A linear operator $T: V \to V$ (where V is finite-dim) is called <u>diagonalizable</u> if $\exists$ an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

A square matrix A is called diagonalizable if L_A is so.

Observation: Say $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

If $\underset{(D_{ij})}{\overset{\|}{D}} = [T]_\beta$ is diagonal, then $\forall\, \vec{v}_j \in \beta$, we have:

$$T(\vec{v}_j) = \sum_{i=1}^{n} D_{ij}\vec{v}_i = \underbrace{D_{jj}}_{\lambda_j}\vec{v}_j = \lambda_j \vec{v}_j$$

Conversely, if $T(\vec{v}_j) = \lambda_j \vec{v}_j$ for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$,

then: $$[T]_\beta = \begin{pmatrix} | & & \\ [T(\vec{v}_1)]_\beta & - & - \\ | & & \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \\ 0 & 0 & & \lambda_n \end{pmatrix}$$

Def: Let T be a linear operator on a vector space V/F.
A non-zero vector $\vec{v} \in V$ is called an <u>eigenvector</u> of T
if $\exists\, \lambda \in F$ s.t. $T(\vec{v}) = \lambda \vec{v}$. In this case, $\lambda \in F$
is called an <u>eigenvalue</u> corresponding to the eigenvector $\vec{v}$.

For a square matrix $A \in M_{n\times n}(F)$, a non-zero vector $\vec{v} \in F^n$
is called an eigenvector of A if it is an eigenvector of L_A.

That is: $\quad A\vec{v} = \lambda \vec{v}$ for some $\lambda \in F$.

λ is called the eigenvalue corresponding to the
eigenvector $\vec{v}$.

<u>Prop:</u> A linear operator $T: V \to V$ (V = fin-dim) is diagonalizable iff $\exists$ an ordered basis β for V consisting of eigenvectors of T.

In such case, if $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then:

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

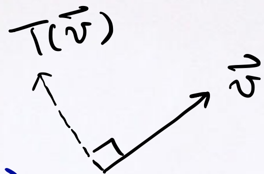
where λ_j is the eigenvalue of T corresponding to $\vec{v}_j$.

Example: • $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $\beta = \underbrace{\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}}_{\text{Check that they are All eigenvectors and } \beta \text{ is basis.}}$

Then: $[L_A]_\beta = \begin{pmatrix} 4 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

• Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $\frac{\pi}{2}$ in counter-clockwise direction.

(Check: $T = L_A$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$)



Then: $T(\vec{v})$ is always perpendicular to $\vec{v}$.

$\therefore$ For $\forall \vec{v} \neq \vec{0}$, it cannot be an eigenvector because:

$T(\vec{v}) \neq \lambda \vec{v}$ for some $\lambda \in F$

Example: Consider $T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by:

Space of smooth function

are infinitely differentiable

$$T(f) = f'$$

Then an eigenvector of T with eigenvalue λ is a non-zero solution of:

$$\frac{df}{dt} = \lambda f(t)$$

$\iff$ $f(t) = Ce^{\lambda t}$ for some constant C.

$\therefore$ all $\lambda \in \mathbb{R}$ is an eigenvalue of T.

Prop: Let $A \in M_{n\times n}(F)$. Then $\lambda \in F$ is an eigenvalue of A iff $\det(A - \lambda I_n) = 0$.

Pf: $\lambda \in F$ is an eigenvalue of A

$\iff \exists \vec{v} \in F^n \setminus \{\vec{0}\}$ s.t. $A\vec{v} = \lambda \vec{v}$.

$\iff (A - \lambda I_n)\vec{v} = \vec{0}$

$\iff A - \lambda I_n$ is singular

$\iff A - \lambda I_n$ is not invertible

$\iff \det(A - \lambda I_n) = 0$.

Def: The characteristic polynomial of $A \in M_{n\times n}(F)$. is defined as the polynomial $f_A(t) \overset{\text{def}}{=} \det(A - tI_n) \in P_n(F)$

Def: Let T be a linear operator on an n-dim vector space V. Choose an ordered basis β for V. Then, the characteristic polynomial of T is defined as the characteristic polynomial of $[T]_\beta$.
(i.e. $f_T(t) \overset{\text{def}}{=} \det([T]_\beta - tI_n) \in P_n(F)$)

__Prop:__ $f_T(t)$ is well-defined, i.e. independent of the choice of β.

__Pf:__ If β' is another ordered basis for V, then:

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q \quad \left(Q = [I_V]_{\beta'}^{\beta} \right)$$

Then:

$$\begin{aligned}
\det([T]_{\beta'} - tI_n) &= \det(Q^{-1}[T]_\beta Q - tI_n) \\
&= \det(Q^{-1}([T]_\beta - tI_n)Q) \\
&= \underbrace{\cancel{\det(Q^{-1})}}_{=\frac{1}{\det(Q)}} \det([T]_\beta - tI_n)\,\cancel{\det(Q)} \\
&= f_T(t).
\end{aligned}$$

Prop: Let $A \in M_{n\times n}(F)$. Then:

- $f_A(t)$ is of degree n and with leading coefficient $(-1)^n$.
- A has at most n distinct eigenvalues.

Pf: Exercise

Def: A polynomial $f(t) \in P(F)$ splits over F if $\exists$ $c, a_1, a_2, \ldots, a_n \in F$ s.t. $f(t) = c(t-a_1)(t-a_2)\cdots(t-a_n)$

Prop: The characteristic polynomial of a diagonalizable linear operator on a finite-dim vector space V/F Splits over F.

Pf: If V is a n-dim and $T: V \to V$ is diagonalizable,

then $\exists$ a basis $\beta \subset V$ s.t.

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

Then: $f_T(t) = \det([T]_\beta - t I_n)$

$$= (-1)^n (t-\lambda_1)(t-\lambda_2) \cdots (t-\lambda_n)$$

Prop: Let T be a linear operator on a vector space V and let λ be an eigenvalue of T. Then, $\vec{v} \in V$ is an eigenvector of T corresponding to λ iff:

$$\vec{v} \in N(T - \lambda I_V) \setminus \{\vec{0}\}$$

Pf: Exercise.

$$T\vec{v} = \lambda \vec{v}$$

$$\Longleftrightarrow (T - \lambda I_V)\vec{v} = \vec{0}$$

Def: Let T be a linear operator on a vector space V and let λ be an eigenvalue of T.

Then: the subspace $E_\lambda \overset{def}{:=} N(T - \lambda I_V) = \{\vec{x} \in V : T(\vec{x}) = \lambda \vec{x}\} \subset V$

is called the <u>eigenspace</u> of T corresponding to λ.

Eigenspaces of a matrix $A \in M_{n \times n}(F)$ is defined as those of L_A

<u>Example</u>: Consider $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$.

$$f_A(t) = \det(A - tI_2) = \det\begin{pmatrix} 1-t & 1 \\ 4 & 1-t \end{pmatrix}$$

$$= t^2 - 2t - 3 = (t-3)(t+1)$$

$\therefore$ The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -1$

For $\lambda_1 = 3$, $B_1 \overset{\text{def}}{=} A - \lambda_1 I_2 = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \overset{\text{RREF}}{\longleftrightarrow} \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$

$$\longleftrightarrow \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$$

$$E_{\lambda_1} = N(B_1) = \left\{ t\begin{pmatrix} 1 \\ 2 \end{pmatrix} : t \in \mathbb{R} \right\}$$

For $\lambda_2 = -1$, $B_2 \overset{\text{def}}{=} A - \lambda_2 I_2 = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$

$$E_{\lambda_2} = N(B_2) = \left\{ t\begin{pmatrix} 1 \\ -2 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Choose $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$ as a basis for $\mathbb{R}^2$.

↑

Basis of eigenvector.

Then: $[L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

$\therefore \; [L_A]_{\substack{\text{standard} \\ \text{ordered} \\ \text{basis}}} = A = Q[L_A]_\beta Q^{-1}$

$\Longleftrightarrow [L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = Q^{-1} A Q$ (Diagonalization of A)