Lecture 1: Vector spaces

The field $F = \mathbb{R}$ (Space of real no.) or $\mathbb{C}$ (Space of complex no.)

<u>Definition:</u> A vector space over F is a set V equipped w/ two operations:

(addition) $+ : V \times V \to V$, $(\underset{\in V}{\vec{x}}, \underset{\in V}{\vec{y}}) \mapsto \vec{x} + \vec{y} \in V$

(Scalar multiplication) $\cdot : F \times V \to V$, $(\underset{\in F}{a}, \underset{\in V}{\vec{x}}) \mapsto a\vec{x} \in V$

satisfying 8 properties:

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(VS1) : $\vec{x} + \vec{y} = \vec{y} + \vec{x} \quad \forall \vec{x}, \vec{y} \in V$

(VS2) : $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z}) \quad \forall \vec{x}, \vec{y}, \vec{z} \in V$

(VS3) : $\exists\, \vec{0} \in V$ s.t. $\vec{x} + \vec{0} = \vec{x} \quad \forall \vec{x} \in V$

(VS4) : $\forall \vec{x} \in V, \ \exists\, \vec{y} \in V$ s.t. $\vec{x} + \vec{y} = \vec{0}$ (inverse)

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(VS5) : $\underset{\underset{F}{\uparrow}}{1}\, \vec{x} = \vec{x} \quad \forall \vec{x} \in V$

(VS6) : $(\underset{\underset{F}{\uparrow}}{a}\, \underset{\underset{F}{\uparrow}}{b})\, \vec{x} = a(b\vec{x}) \quad \forall a, b \in F, \ \forall \vec{x} \in V$

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(VS7) : $\underset{\underset{F}{\uparrow}}{a}\, (\underset{\underset{V}{\uparrow}}{\vec{x}} + \underset{\underset{V}{\uparrow}}{\vec{y}}) = a\vec{x} + a\vec{y} \quad \forall a \in F, \ \forall \vec{x}, \vec{y} \in V$

(VS8) : $(a + b)\vec{x} = a\vec{x} + b\vec{x} \quad \forall a, b \in F, \ \forall \vec{x} \in V$

<u>Remark</u>: an element in F is called <u>scalar</u>.

an element in V is called <u>vector</u>.

Examples:

- $F^n = \{(x_1, x_2, \dots, x_n) : x_j \in F \text{ for } j = 1, 2, \dots, n\}$ w/

$(x_1, x_2, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$

$a(x_1, \dots, x_n) = (ax_1, ax_2, \dots, ax_n)$

- $M_{m \times n}(F) = \{ m \times n$ matrices w/ entries in $F \}$
w/ matrix addition and scalar multiplication

- $P(F) = \{$ polynomials w/ coefficients in $F \}$
w/ polynomial addition and scalar multiplication.

- $F^\infty = \{(x_1, x_2, \dots) : x_j \in F, j = 1, 2, \dots\}$
w/ component-wise addition and scalar multiplication.

- $\mathrm{Sym}_{n\times n}(F) = \{ n \times n$ symmetric matrices A w/ entries in F : $A^T = A \}$

- Let S be any non-empty set.

  Then: $\mathcal{F}(S, F) = \{$ functions $f : S \to F \}$

  is a vector space over F under:

$$(\underset{\in \mathcal{F}(S,F)}{f} + \underset{\in \mathcal{F}(S,F)}{g})(\overset{\in S}{s}) \overset{\text{def}}{=} f(s) + g(s) \; ; \qquad (\underset{\in F}{a} f)(s) \overset{\text{def}}{=} a f(s).$$

- $\mathbb{C}$ is a vector space over $F = \mathbb{C}$

<u>Remark</u>: $V = \mathbb{R}$ is NOT a vector space over $F = \mathbb{C}$.