MATH2040A/B Homework 7 Solution

(Sec 5.1 Q2(c))

$$T\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}0\\-1\\-1\end{pmatrix}=-1\begin{pmatrix}0\\1\\1\end{pmatrix},\ T\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}=1\begin{pmatrix}1\\-1\\0\end{pmatrix},\ T\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}-1\\0\\-2\end{pmatrix}-1\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

Hence all vectors in β are eigenvectors and

$$[T]_\beta=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

(Sec 5.1 Q2(f)) Ans : We have

$$T\begin{pmatrix}1&0\\1&0\end{pmatrix}=-3\begin{pmatrix}1&0\\1&0\end{pmatrix},T\begin{pmatrix}-1&2\\0&0\end{pmatrix}=\begin{pmatrix}-1&2\\0&0\end{pmatrix},T\begin{pmatrix}1&0\\2&0\end{pmatrix}=\begin{pmatrix}1&0\\2&0\end{pmatrix},T\begin{pmatrix}-1&0\\0&2\end{pmatrix}=\begin{pmatrix}-1&0\\0&2\end{pmatrix}$$

Hence all vectors in β are eigenvectors and

$$[T]_\beta=\begin{pmatrix}-3&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

(Sec 5.1 Q3(c)) Ans : $\det(A-\lambda I)=-(i-\lambda)(i+\lambda)-2$, solving gives $\lambda=1,-1$. For $\lambda=1$, $A\sim\begin{pmatrix}1&-0.5-0.5i\\0&0\end{pmatrix}$ Hence an eigenvector of A is $(0.5+0.5i,1)^T$.

For $\lambda=-1$, $A\sim\begin{pmatrix}1&0.5-0.5i\\0&0\end{pmatrix}$ hence an eigenvector of A is $(-0.5+0.5i,1)^T$. Together, letting $Q:=\begin{pmatrix}0.5+0.5i&-0.5+0.5i\\1&1\end{pmatrix}$, we have $Q^{-1}AQ=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$.

(Sec 5.1 Q3(d)) Ans : $\det(A-\lambda I)=-\lambda(1-\lambda)^2$. We have eigenvalues $\lambda=0,\lambda=1$. For $\lambda=0$, $A\sim\begin{pmatrix}1&0&-1/2\\0&1&-2\\0&0&0\end{pmatrix}$ hence $(1/2,2,1)$ is an eigenvector.

When $\lambda=1$, $A\sim\begin{pmatrix}1&0&-1\\0&0&0\\0&0&0\end{pmatrix}$, hence $(0,1,0),(1,0,1)$ are two eigenvectors.

Setting $Q=\begin{pmatrix}1/2&0&1\\2&1&0\\1&0&1\end{pmatrix}$, we have $Q^{-1}AQ=\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

(Sec 5.1 Q4(g)) Ans: We consider $\alpha:=\{1,x,x^2,x^3\}$ the standard basis. Then

$$[T]_\alpha\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\0\\0\end{pmatrix},[T]_\alpha\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}-2\\1\\0\\0\end{pmatrix},[T]_\alpha\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}-2\\0\\2\\0\end{pmatrix},[T]_\alpha\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}-8\\6\\0\\3\end{pmatrix}$$

and hence

$$[T]_\alpha=\begin{pmatrix}-1&-2&-2&-8\\0&1&0&6\\0&0&2&0\\0&0&0&3\end{pmatrix}$$

Hence the eigenvalues are $-1, 1, 2, 3$.
When $\lambda = -1$, We can get that $(1, 0, 0, 0)^T$ is an eigenvector with eigenvalue -1 by solving the linear equation $(-I - [T]_\alpha)x = 0$.(Any non-all-zeros solution of the equation is an eigenvector with eigenvalue -1).

When $\lambda = 1$, similarly we can get that $(-1, 1, 0, 0)^T$ is an eigenvector with 1 eigenvalue 1

When $\lambda = 2$, similarly, we can get that $(-2, 0, 3, 0)^T$ is an eigenvector with 1 eigenvalue 2

When $\lambda = 3$ similarly, we can get that $(-7, 6, 0, 2)^T$ is an eigenvector with 1 eigenvalue 3
Let $\beta = \{1, -1 + x, -2 + 3x^2, -7 + 6x + 2x^3\}$, then

$$[T]_\beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(Sec 5.1 Q4(h)) Ans: We simply observe that $T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,$T\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ So the basis is $\beta = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$.

(Sec 5.1 Q6) Ans: We prove another statement: if $V \cong V'$, $\phi : V \to V'$ an isomorphism, $T : V \to V$ a linear operator on V, then if $Tv = \lambda v$,

$$(\phi \circ T \circ \phi^{-1})\phi(v) = \lambda\phi(v)$$

$(\phi \circ T \circ \phi^{-1})\phi(v) = \phi \circ Tv = \lambda\phi(v)$
Now take $V' = \mathbb{F}^{\dim V}$, β a basis of V' and consider $\phi(v) = [v]_\beta$.

(Sec 5.1 Q8) (a) T is invertible if and only if $\det(T) \neq 0$ if and only if $\det(T - 0I) \neq 0$ if and only if 0 is not an eigenvalue of T.

(b) From (a), eigenvalues are not zero. Suffices to show one way since T, T^{-1} are inverse to each other.

$$Tv = \lambda v$$
$$1\lambda v = T^{-1}v$$

so λ^{-1} is an eigenvalue of T^{-1}.

(c) 1. Statement:$A \in F^{n\times n}$ is invertible if and only if 0 is not an eigenvalue of A.
Proof: Consider $T = L_A$.
2. Statement:$A \in F^{n\times n}$ is invertible. A Scalar λ is an an eigenvalue of A if and only if λ^{-1} is an an eigenvalue of A^{-1}.
Proof: Consider $T = L_A$.

(Sec 5.1 Q15) (a) $T^m(x) = T^{m-1}(Tx) = T^{m-1}(\lambda x) = \lambda T^{m-1}(x)$.
Repeating it $m - 1$ times, we get $T^m(x) = \lambda^m x$.

(b) Let $A \in F^{n\times n}$, and let x be an eigenvector of A corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of A^m corresponding to the eigenvalue λ^m .
Proof: Consider $T = L_A$ and using $(L_A)^k = L_{A^k}$ for $\forall k \in N^+$.

(Sec 5.1 Q20) Ans: $\det(A - tI) = f(t)$, hence $a_0 = f(0) = \det(A)$. Hence $a_0 \neq 0$ if and only if $\det(A) \neq 0$ if and only if A invertible.

(Sec 5.1 Q22) (a) We assume $g(t) = \sum_{k=0}^n a_k x^k$. Using the conclusion of Q15 Sec 5.1, $g(T)(x) = \sum_{k=0}^n a_k T^k(x) = \sum_{k=0}^n a_k(\lambda^k x) = (\sum_{k=0}^n a_k\lambda^k)x = g(\lambda)x$.

(b) Let $A \in F^{n\times n}$, and let $g(t)$ be a polynomial with coefficients from F. If x is an eigenvector of A corresponding to the eigenvalue λ, then $g(T)(x) = g(\lambda)x$.

Proof: Consider $T = L_A$ and using the results of Q15.

(c) We have $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix} = 4 \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Then $g(A) = 2 \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 10 \\ 15 & 19 \end{pmatrix}$ and $g(\lambda) = g(4) = 2 \cdot 4^2 - 4 + 1 = 29$.

Hence $g(A)(x) = \begin{pmatrix} 14 & 10 \\ 15 & 19 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 58 \\ 87 \end{pmatrix} = 29 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = g(\lambda)x$