MATH2040A/B Homework 2 Solution 6

(Sec 2.4 Q2) Ans: (d) No. They have different dimension 4 and 3.

(e) No. They have different dimension 4 and 3.

(Sec 2.4 Q7) Ans: (a) If A is invertible, then A^{-1} exsits and $A = A^{-1}A^2 = A^{-1}O = O$, but O is not invertible, so this is a contradiction and A is not invertible.

(b) If A is invertible, then $B = A^{-1}AB = A^{-1}O = O$, so there is no such nonzero matrix B and A is not invertible.

(Sec 2.4 Q15) Ans: Write $\beta = \{u_1, ..., u_n\}$, where $u_1, ..., u_n$ are distinct vectors in V.
($\Rightarrow$) Suppose T is an isomorphism. Then $T(u_1), ..., T(u_n)$ are distinct. Suppose $a_1, ..., a_n$ are scalars such that

$$a_1T(u_1) + \cdots + a_nT(u_n) = \vec{0}.$$

Then

$$a_1u_1 + \cdots + a_nu_n = T^{-1}(a_1T(u_1) + \cdots + a_nT(u_n)) = T^{-1}(\vec{0}) = \vec{0}.$$

As β is a basis for V and in particular linearly independent, $a_1 = \cdots = a_n = 0$. Thus, $T(\beta) = \{T(u_1), ..., T(u_n)\}$ is also linearly independent.
Since $\dim W = n$ and the cardinality of $T(\beta)$ is also n, $T(\beta)$ is a basis for V.
(**Alternatively**, suppose $w \in W$. Then $\exists$ scalars $a_1, ..., a_n$ such that $T^{-1}(w) = \sum_{i=1}^{n} a_iu_i$ and hence $w = T(T^{-1}(w)) = \sum_{i=1}^{n} a_iT(u_i) \in \operatorname{span} T(\beta)$. Thus, $T(\beta)$ spans W. To conclude, $T(\beta)$ is a basis for W.)

(Sec 2.4 Q16) Ans: $\forall A, A' \in \mathsf{M}_{n\times n}(F)$ and $\forall a \in F$,

$$\begin{aligned}\Phi(A + A') &= B^{-1}(A + A')B = B^{-1}AB + B^{-1}A'B = \Phi(A) + \Phi(A')\\ \Phi(aA) &= B^{-1}(aA)B = a(B^{-1}AB) = a\Phi(A).\end{aligned}$$

Hence, Φ is a linear transformation.
Suppose $A \in \mathsf{M}_{n\times n}(F)$ and $\Phi(A)$ is the $n \times n$ zero matrix O over F. Then $A = B\Phi(A)B^{-1} = BOB^{-1} = O$. Hence, $\Phi : \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ is one-to-one.
$\forall C \in \mathsf{M}_{n\times n}(F)$, $BCB^{-1} \in \mathsf{M}_{n\times n}(F)$ and $\Phi(BCB^{-1}) = B(B^{-1}CB)B^{-1} = C$. Hence, $\Phi : \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ is also onto. Therefore, Φ is an isomorphism.

(Sec 2.4 Q17) Ans: (a) If $y_1, y_2 \in T(V_0)$ and $y_1 = T(x_1)$, $y_2 = T(x_2)$, we have that $y_1 + y_2 = T(x_1 + x_2) \in T(V_0)$ and $cy_1 = T(cx_1) = T(V_0)$. Finally since V_0 is a subspace and so $0 = T(0) \in T(V_0)$, $T(V_0)$ is a subspace of W.

(b) We can consider a mapping T' from V_0 to $T(V_0)$ by $T'(x) = T(x)$ for all $x \in V_0$. It's natural that T' is surjective. And it's also injective since T is injective. So by Dimension Theorem we have that $\dim(V_0) = \dim(N(T')) + \dim(R(T')) = \dim(T(V_0))$

(Sec 2.5 Q2) Ans: (b) $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$

(Sec 2.5 Q3) Ans: (a) $\begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}$

(Sec 2.5 Q4) Ans: We first find out the change of coordinate matrix Q that changes β'-coordinates into β-coordinates, which is

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Note that we have

$$[T]_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}.$$

Now, by Theorem 2.23 in Sec. 2.5,

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}.$$

(Sec 2.5 Q6) Ans: Let α be the standard basis (of $\mathbb{F}^2$ or $\mathbb{F}^3$). We have that $A = [L_A]_\alpha$ and hence $[L_A]_\beta = [I]_\alpha^\beta [L_A]_\alpha [I]_\beta^\alpha$. So now we can calculate $[L_A]_\beta$ and $Q = [I]_\beta^\alpha$ abd $Q^{-1} = [I]_\alpha^\beta$.

(b) $[L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(c) $[L_A]_\beta = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

(Sec 2.5 Q7) Ans: We may let β be the standard basis and $\alpha = \{(1, m), (-m, 1)\}$ be another basis for $\mathbb{R}^2$.

(a) We have that $[T]_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Q^{-1} = [I]_\alpha^\beta = \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}$. We also can calculate that $Q = [I]_\beta^\alpha = \begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ -\frac{m}{m^2+1} & \frac{1}{m^2+1} \end{pmatrix}$. So finally we get

$$[T]_\beta = Q^{-1}[T]_\alpha Q = \begin{pmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{pmatrix}.$$

That is, $T(x, y) = (\frac{x+2ym-xm^2}{m^2+1}, \frac{-y+2xm+ym^2}{m^2+1})$.

(b) Simalarly we have that $[T]_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. And with the same Q and Q^{-1} we get

$$[T]_\beta = Q^{-1}[T]_\alpha Q = \begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ \frac{m}{m^2+1} & \frac{m^2}{m^2+1} \end{pmatrix}.$$

That is, $T(x, y) = (\frac{x+ym}{m^2+1}, \frac{xm+ym^2}{m^2+1})$.