MMAT5390 Mathematical Image Processing
Practice Midterm Examination Solution

1. Note that H is a 4×4 matrix; hence it represents a linear transformation on 2×2 images.

   H is not block-circulant. For example, consider the $y = 1, \beta = 1$-submatrix of H, i.e. $\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$. This is not a circulant matrix, as the shift-operator T maps $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ instead of $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$. Hence h is not shift-invariant with h_s being 2-periodic in both arguments.
   (However, H is block-Toeplitz and thus h is shift-invariant.)

2. Assume f and H are periodically extended.

   (a) $H * f$ is the 5×5 matrix whose entries are given by

   $$H * f(\alpha, \beta) = \sum_{m=-2}^{2} \sum_{n=-2}^{2} H(m,n) f(\alpha - m, \beta - n).$$

   (b)

   $$\begin{aligned} H * f(\alpha, \beta) &= \sum_{m=-2}^{2} \sum_{n=-2}^{2} H(m,n) f(\alpha - m, \beta - n) \\ &= \sum_{m=-2}^{2} \sum_{n=-2}^{2} a_{m+3} b_{n+3} f(\alpha - m, \beta - n) \\ &= \sum_{m=-2}^{2} \sum_{n=-2}^{2} H_1(m,0) H_2(0,n) f(\alpha - m, \beta - n) \\ &= \sum_{n=-2}^{2} H_2(0,n) \sum_{m=-2}^{2} \sum_{n'=-2}^{2} H_1(m,n') f(\alpha - m, \beta - n - n') \\ &= \sum_{n=-2}^{2} H_2(0,n) H_1 * f(\alpha, \beta - n) \end{aligned}$$

   $$\begin{aligned} &= \sum_{m'=-2}^{2} \sum_{n=-2}^{2} H_2(m', n) H_1 * f(\alpha - m', \beta - n) \\ &= H_2 * (H_1 * f)(\alpha, \beta). \end{aligned}$$

   Hence $H * f = H_1 * (H_2 * f)$.

3. (a) Assume $I_1, I_2 \in \mathcal{I}$ are periodically extended.
   The discrete convolution $I_1 * I_2$ of I_1 and I_2 is the $(2N+1) \times (2N+1)$ matrix whose entries are defined by

   $$I_1 * I_2(\alpha, \beta) = \sum_{m=-N}^{N} \sum_{n=-N}^{N} I_1(m,n) I_2(\alpha - m, \beta - n).$$

Let $I_1, I_2 \in \mathcal{I}$, and let $c \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{O}(I_1 + cI_2) &= (I_1 + cI_2) * H \\
&= \left[\sum_{m=-N}^{N} \sum_{n=-N}^{N} (I_1 + cI_2)(m,n) H(\alpha - m, \beta - n) \right]_{-N \le \alpha, \beta \le N} \\
&= \left[\sum_{m=-N}^{N} \sum_{n=-N}^{N} [I_1(m,n)H(\alpha - m, \beta - n) + cI_2(m,n)H(\alpha - m, \beta - n)] \right]_{-N \le \alpha, \beta \le N} \\
&= [I_1 * H(\alpha, \beta) + cI_2 * H(\alpha, \beta)]_{-N \le \alpha, \beta \le N} \\
&= I_1 * H + cI_2 * H \\
&= \mathcal{O}(I_1) + c\mathcal{O}(I_2).
\end{aligned}$$

Hence $\mathcal{O}$ is linear.

For any $I \in \mathcal{I}$, the PSF h of $\mathcal{O}$ satisfies

$$\sum_{x=-N}^{N} \sum_{y=-N}^{N} h(x, \alpha, y, \beta) I(x, y) = I * H(\alpha, \beta) = \sum_{m=-N}^{N} \sum_{n=-N}^{N} I(m,n) H(\alpha - m, \beta - n),$$

and thus $h(x, \alpha, y, \beta) = H(\alpha - x, \beta - y)$.

Hence h is shift-invariant.

(b) Let $H_1, H_2 \in \mathcal{I}$. Then

$$\begin{aligned}
I * (H_1 * H_2)(\alpha, \beta) &= \sum_{m=-N}^{N} \sum_{n=-N}^{N} I(m,n) H_1 * H_2(\alpha - m, \beta - n) \\
&= \sum_{m=-N}^{N} \sum_{n=-N}^{N} I(m,n) \sum_{m'=-N}^{N} \sum_{n'=-N}^{N} H_1(m', n') H_2(\alpha - m - m', \beta - n - n')
\end{aligned}$$

$$= \sum_{m=-N}^{N} \sum_{n=-N}^{N} I(m,n) \sum_{m''=-N+m}^{N+m} \sum_{n''=-N+n}^{N+n} H_1(m''-m, n''-n) H_2(\alpha - m'', \beta - n'')$$

$$= \begin{cases} \sum_{m=-N}^{N} \sum_{n=-N}^{N} I(m,n) & \left(\sum_{m''=-N+m}^{-N-1} + \sum_{m''=-N}^{N+m} \right) \sum_{n''=-N+n}^{N+n} \\ & H_1(m''-m, n''-n) H_2(\alpha - m'', \beta - n'') \text{ if } -N \le m \le 0 \\ \sum_{m=-N}^{N} \sum_{n=-N}^{N} I(m,n) & \left(\sum_{m''=-N+m}^{N} + \sum_{m''=N+1}^{N+m} \right) \sum_{n''=-N+n}^{N+n} \\ & H_1(m''-m, n''-n) H_2(\alpha - m'', \beta - n'') \text{ if } 1 \le m \le N \end{cases}$$

$$= \sum_{m=-N}^{N} \sum_{n=-N}^{N} I(m,n) \sum_{m''=-N}^{N} \sum_{n''=-N+n}^{N+n} H_1(m''-m, n''-n) H_2(\alpha - m'', \beta - n'')$$

$$= \begin{cases} \sum_{m=-N}^{N} \sum_{n=-N}^{N} I(m,n) \sum_{m''=-N}^{N} & \left(\sum_{n''=-N+n}^{-N-1} + \sum_{n''=-N}^{N+n} \right) \\ & H_1(m''-m, n''-n) H_2(\alpha - m'', \beta - n'') \text{ if } -N \le n \le 0 \\ \sum_{m=-N}^{N} \sum_{n=-N}^{N} I(m,n) \sum_{m''=-N}^{N} & \left(\sum_{n''=-N+n}^{N} + \sum_{n''=N+1}^{N+n} \right) \\ & H_1(m''-m, n''-n) H_2(\alpha - m'', \beta - n'') \text{ if } 1 \le n \le N \end{cases}$$

$$= \sum_{m=-N}^{N} \sum_{n=-N}^{N} I(m,n) \sum_{m''=-N}^{N} \sum_{n''=-N}^{N} H_1(m''-m, n''-n) H_2(\alpha - m'', \beta - n'')$$

$$= \sum_{m''=-N}^{N} \sum_{n''=-N}^{N} I * H_1(m'', n'') H_2(\alpha - m'', \beta - n'')$$

$$= (I * H_1) * H_2(\alpha, \beta).$$

Hence $I * (H_1 * H_2) = (I * H_1) * H_2$.

(c)

$$I * H(\alpha, \beta) = \sum_{m=-N}^{N} \sum_{n=-N}^{N} I(m,n) H(\alpha - m, \beta - n)$$

$$= \sum_{m'=\alpha-N}^{\alpha+N} \sum_{n'=\beta-N}^{\beta+N} I(\alpha - m', \beta - n') H(m', n')$$

$$= \begin{cases} \left(\sum_{m'=\alpha-N}^{-N-1} + \sum_{m'=-N}^{\alpha+N} \right) \sum_{n'=\beta-N}^{\beta+N} I(\alpha - m', \beta - n') H(m', n') & \text{if } -N \le \alpha \le 0 \\ \left(\sum_{m'=\alpha-N}^{N} + \sum_{m'=N+1}^{\alpha+N} \right) \sum_{n'=\beta-N}^{\beta+N} I(\alpha - m', \beta - n') H(m', n') & \text{if } 1 \le \alpha \le N \end{cases}$$

$$= \sum_{m'=-N}^{N} \sum_{n'=\beta-N}^{\beta+N} H(m', n') I(\alpha - m', \beta - n')$$

$$= \begin{cases} \sum_{m'=-N}^{N} \left(\sum_{n'=\beta-N}^{-N-1} + \sum_{n'=-N}^{\beta+N} \right) H(m',n')I(\alpha-m',\beta-n') & \text{if } -N \le \beta \le 0 \\ \sum_{m'=-N}^{N} \left(\sum_{n'=\beta-N}^{N} + \sum_{n'=N+1}^{\beta+N} \right) H(m',n')I(\alpha-m',\beta-n') & \text{if } 1 \le \beta \le N \end{cases}$$

$$= \sum_{m'=-N}^{N} \sum_{n'=-N}^{N} H(m',n')I(\alpha-m',\beta-n')$$

$$= H * I(\alpha,\beta).$$

Hence $I * H = H * I$.

4. (a) Note that $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 7 & 3 \\ 5 & 7 \end{pmatrix}$ are Toeplitz, and that H_1 is circulant (hence Toeplitz) when viewed as a matrix of 2×2 blocks.
Hence H_1 is block-Toeplitz, and thus represents a shift-invariant linear transformation on 2×2 images.
On the other hand, as $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$ is not circulant, H_1 is not block-circulant. Hence h_s is not 2-periodic in some of its arguments.

(b) Note that H_2 is a 9×9 matrix; hence it represents a linear transformation on 3×3 images.
H_2 is block-circulant. The $y = 1, \beta = 1$-, the $y = 2, \beta = 2$- and the $y = 3, \beta = 3$-submatrices of H_2 are all $\begin{pmatrix} 9 & 9 & 18 \\ 18 & 9 & 9 \\ 9 & 18 & 9 \end{pmatrix}$, which is circulant; the $y = 2, \beta = 1$-, the $y = 3, \beta = 2$- and the $y = 1, \beta = 3$-submatrices of H_2 are all $\begin{pmatrix} 9 & 9 & 18 \\ 18 & 9 & 9 \\ 9 & 18 & 9 \end{pmatrix}$, which is circulant; the $y = 3, \beta = 1$-, the $y = 1, \beta = 2$- and the $y = 2, \beta = 3$-submatrices of H_2 are all $\begin{pmatrix} 18 & 18 & 36 \\ 36 & 18 & 18 \\ 18 & 36 & 18 \end{pmatrix}$, which is also circulant. Hence h is shift-invariant with h_s being 3-periodic in both arguments.

5. Suppose A is symmetric but not diagonalizable. Then there is a generalized eigenvector v ,that is not an eigenvector, with order m associated with an eigenvalue λ such that

$$(A - \lambda I)^m v = 0$$
$$(A - \lambda I)^k v = 0 \qquad ,1 \le k < m$$

The above is not very easy to prove. A lot of linear algebra textbooks put it as a fact without proof. If one is really interested in the proof, he/she should refer to Section 4.7 of Michael Artin's algebra textbook. Here we omit the proof. Let l to be the least integer that is greater than or equal to $m/2$. Then we have the following contradiction:

$$0 = v^*(A - \lambda I)^{2l} v = v^*(A - \lambda I)^l (A - \lambda I)^l v = \|(A - \lambda I)^l v\|^2 \neq 0$$

Clearly this question is too difficult for exam. Please just take it as a reference.

6. The following two facts are obvious:

(a) Trace of a matrix is equal to the sum of its eigenvalues.

(b) If λ is an eigenvalue of A, λ^n is an eigenvalue of A^n

Combining the two facts, we have the statement.

7. (a) $ff^T = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}$.

For $\lambda = 20$:

$$\left[\begin{array}{cc|c} -10 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

which gives unit eigenvector $\vec{u}_1 = (0,1)^T$.

For $\lambda = 10$:

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -10 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

which gives unit eigenvector $\vec{u}_2 = (1,0)^T$.

Then $\vec{v}_1 = \frac{f^T\vec{u}_1}{\sigma_1} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}(0,2,0,1)^T$, and

$\vec{v}_2 = \frac{f^T\vec{u}_2}{\sigma_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{10}}(1,0,3,0)^T$.

$f^T f = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 16 & 0 & 8 \\ 3 & 0 & 9 & 0 \\ 0 & 8 & 0 & 4 \end{pmatrix}$.

For $A^T A\vec{v} = 0$,

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 16 & 0 & 8 & 0 \\ 3 & 0 & 9 & 0 & 0 \\ 0 & 8 & 0 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

which gives orthonormal eigenvectors $\vec{v}_3 = \frac{1}{\sqrt{10}}(-3,0,1,0)^T$ and $\vec{v}_4 = \frac{1}{\sqrt{5}}(0,1,0,-2)^T$.

Hence an SVD of A is $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 2\sqrt{5} & 0 & 0 & 0 \\ 0 & \sqrt{10} & 0 & 0 \end{pmatrix} \text{ and } V = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 & 1 & -3 & 0 \\ 2\sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & 3 & 1 & 0 \\ \sqrt{2} & 0 & 0 & -2\sqrt{2} \end{pmatrix}.$$

(b) The eigenimages are given by

$\vec{u}_1\vec{v}_1^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{pmatrix}$ and

$\vec{u}_2\vec{v}_2^T = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{10}}{10} & 0 & \frac{3\sqrt{10}}{10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Hence $A = 2\sqrt{5} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{pmatrix} + \sqrt{10} \begin{pmatrix} \frac{\sqrt{10}}{10} & 0 & \frac{3\sqrt{10}}{10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

8. (a) $\tilde{H} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$.

$$
\begin{aligned}
f_{\text{Haar}} &= \tilde{H} f \tilde{H}^T \\
&= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 4 & 6 & 6 \\ 6 & 1 & 6 & 3 \\ 1 & 2 & 1 & 5 \\ 6 & 4 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 18 & 11 & 19 & 15 \\ 4 & -1 & 5 & 3 \\ -\sqrt{2} & 3\sqrt{2} & 0 & 3\sqrt{2} \\ -5\sqrt{2} & -2\sqrt{2} & -5\sqrt{2} & 4\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 63 & -5 & 7\sqrt{2} & 4\sqrt{2} \\ 11 & -5 & 5\sqrt{2} & 2\sqrt{2} \\ 5\sqrt{2} & -\sqrt{2} & -8 & -6 \\ -8\sqrt{2} & -6\sqrt{2} & -6 & -18 \end{pmatrix} = \begin{pmatrix} \frac{63}{4} & -\frac{5}{4} & \frac{7\sqrt{2}}{4} & \sqrt{2} \\ \frac{11}{4} & -\frac{5}{4} & \frac{5\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -2 & -\frac{3}{2} \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & -\frac{3}{2} & -\frac{9}{2} \end{pmatrix}.
\end{aligned}
$$

(b) Since $\tilde{H}$ is unitary, for any $g \in M_{4\times 4}(\mathbb{R})$,

$$\|\tilde{H}^T g \tilde{H} - f\|_F = \|\tilde{H}^T (g - f_{\text{Haar}})\tilde{H}\|_F = \|g - f_{\text{Haar}}\|_F.$$

Hence one should choose to discard the entries with smaller absolute values so as to minimize the Frobenius norm of the difference.

Hence the matrix that should be kept is either $f'_{\text{Haar}} = \begin{pmatrix} \frac{63}{4} & 0 & \frac{7\sqrt{2}}{4} & 0 \\ \frac{11}{4} & 0 & \frac{5\sqrt{2}}{4} & 0 \\ \frac{5\sqrt{2}}{4} & 0 & -2 & 0 \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & -\frac{3}{2} & -\frac{9}{2} \end{pmatrix}$,

whose reconstructed image is given by $\tilde{H}^T f'_{\text{Haar}} \tilde{H}$

$$
\begin{aligned}
&= \frac{1}{16}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 63 & 0 & 7\sqrt{2} & 0 \\ 11 & 0 & 5\sqrt{2} & 0 \\ 5\sqrt{2} & 0 & -8 & 0 \\ -8\sqrt{2} & -6\sqrt{2} & -6 & -18 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix} 84 & 0 & 4\sqrt{2} & 0 \\ 64 & 0 & 20\sqrt{2} & 0 \\ 36 & -12 & -4\sqrt{2} & -18\sqrt{2} \\ 68 & 12 & 8\sqrt{2} & 18\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix} 92 & 76 & 84 & 84 \\ 104 & 24 & 64 & 64 \\ 16 & 32 & 12 & 84 \\ 96 & 64 & 92 & 20 \end{pmatrix} = \begin{pmatrix} \frac{23}{4} & \frac{19}{4} & \frac{21}{4} & \frac{21}{4} \\ \frac{13}{2} & \frac{3}{2} & 4 & 4 \\ 1 & 2 & \frac{3}{4} & \frac{21}{4} \\ 6 & 4 & \frac{23}{4} & \frac{5}{4} \end{pmatrix};
\end{aligned}
$$

or keep $f''_{\text{Haar}} = \begin{pmatrix} \frac{63}{4} & 0 & \frac{7\sqrt{2}}{4} & 0 \\ \frac{11}{4} & 0 & \frac{5\sqrt{2}}{4} & 0 \\ \frac{5\sqrt{2}}{4} & 0 & -2 & -\frac{3}{2} \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & 0 & -\frac{9}{2} \end{pmatrix}$,

whose reconstructed image is given by $\tilde{H}^T f''_{\text{Haar}} \tilde{H}$

$$
= \frac{1}{16}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 63 & 0 & 7\sqrt{2} & 0 \\ 11 & 0 & 5\sqrt{2} & 0 \\ 5\sqrt{2} & 0 & -8 & -6 \\ -8\sqrt{2} & -6\sqrt{2} & 0 & -18 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}
$$

$$
= \frac{1}{16}\begin{pmatrix} 84 & 0 & 4\sqrt{2} & -6\sqrt{2} \\ 64 & 0 & 20\sqrt{2} & 6\sqrt{2} \\ 36 & -12 & 2\sqrt{2} & -18\sqrt{2} \\ 68 & 12 & 2\sqrt{2} & 18\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}
$$

$$
= \frac{1}{16}\begin{pmatrix} 92 & 76 & 72 & 96 \\ 104 & 24 & 76 & 52 \\ 28 & 20 & 12 & 84 \\ 84 & 76 & 92 & 20 \end{pmatrix} = \begin{pmatrix} \frac{23}{4} & \frac{19}{4} & \frac{9}{2} & 6 \\ \frac{13}{2} & \frac{3}{2} & \frac{19}{4} & \frac{13}{4} \\ \frac{7}{4} & \frac{5}{4} & \frac{3}{4} & \frac{21}{4} \\ \frac{21}{4} & \frac{19}{4} & \frac{23}{4} & \frac{5}{4} \end{pmatrix}.
$$

9. (a) $\tilde{W} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$.

$$
f_{\text{Walsh}} = \tilde{W} f \tilde{W}^T
$$

$$
= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2 & 4 & 4 \\ 4 & -2 & 4 & 1 \\ -2 & -1 & -1 & 3 \\ 4 & 1 & 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}
$$

$$
= \frac{1}{4}\begin{pmatrix} 9 & 0 & 11 & 6 \\ -5 & 0 & -5 & -4 \\ -5 & -6 & -5 & 2 \\ -7 & 2 & -5 & 8 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}
$$

$$
= \frac{1}{4}\begin{pmatrix} 26 & 8 & -4 & 14 \\ -14 & -4 & 4 & -6 \\ -14 & 8 & -8 & -6 \\ -2 & 8 & -4 & -22 \end{pmatrix} = \begin{pmatrix} \frac{13}{2} & 2 & -1 & \frac{7}{2} \\ -\frac{7}{2} & -1 & 1 & -\frac{3}{2} \\ -\frac{7}{2} & 2 & -2 & -\frac{3}{2} \\ -\frac{1}{2} & 2 & -1 & -\frac{11}{2} \end{pmatrix}.
$$

(b) The modified Walsh transform f'_{Walsh} is $\begin{pmatrix} 7 & 2 & -1 & 4 \\ -4 & -1 & 1 & -2 \\ -4 & 2 & -2 & -2 \\ -1 & 2 & -1 & -6 \end{pmatrix}$, whose reconstructed image is given by

$$
\tilde{W}^T f'_{\text{Walsh}} \tilde{W} = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 7 & 2 & -1 & 4 \\ -4 & -1 & 1 & -2 \\ -4 & 2 & -2 & -2 \\ -1 & 2 & -1 & -6 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}
$$

$$
= \frac{1}{4}\begin{pmatrix} 14 & 3 & -1 & 2 \\ 8 & 3 & -3 & 10 \\ -2 & 5 & -3 & -6 \\ 8 & -3 & 3 & 10 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}
$$

$$
= \frac{1}{4}\begin{pmatrix} 14 & 8 & 18 & 16 \\ 18 & -8 & 18 & 4 \\ -10 & -4 & -6 & 12 \\ 18 & 4 & 18 & -8 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} & 2 & \frac{9}{2} & 4 \\ \frac{9}{2} & -2 & \frac{9}{2} & 1 \\ -\frac{5}{2} & -1 & -\frac{3}{2} & 3 \\ \frac{9}{2} & 1 & \frac{9}{2} & -2 \end{pmatrix}
$$

10. (a) Note that $W_0 = \mathbf{1}_{[0,1)}$ and thus $(W_0)^2 = \mathbf{1}_{[0,1)}$. Recall that for any $n \in \mathbb{N} \cup \{0\}$, W_n is defined by the recursive relation:

$$W_{2j+q}(t) = (-1)^{\lfloor \frac{j}{2} \rfloor + q} W_j(2t) + (-1)^{j+\lfloor \frac{j}{2} \rfloor} W_j(2t-1)$$

for $j \in \mathbb{N} \cup \{0\}$ and $q \in \{0, 1\}$.
Hence for any $n \in \mathbb{N}$, $(W_n)^2 \equiv \mathbf{1}_{[0,1)}$ and thus

$$\int_{\mathbb{R}} [W_n(t)]^2 \, dt = \int_0^1 \, dt = 1.$$

(b) i. Suppose $j_1 = j_2$. Then $m_1 = 2j_1$ and $m_2 = 2j_1 + 1$, and

$$\begin{aligned}
\langle W_{m_1}, W_{m_2} \rangle &= \int_{\mathbb{R}} W_{2j_1}(t) W_{2j_1+1}(t) \, dt \\
&= \int_0^{\frac{1}{2}} (-1)^{\lfloor \frac{j_1}{2} \rfloor} W_{j_1}(2t) \cdot (-1)^{\lfloor \frac{j_1}{2} \rfloor + 1} W_{j_1}(2t) \, dt \\
&+ \int_{\frac{1}{2}}^1 (-1)^{j_1 + \lfloor \frac{j_1}{2} \rfloor} W_{j_1}(2t-1) \cdot (-1)^{j_1 + \lfloor \frac{j_1}{2} \rfloor} W_{j_1}(2t-1) \, dt \\
&= -\int_0^1 [W_{j_1}(u)]^2 \, d(\frac{u}{2}) + \int_0^1 [W_{j_1}(v)]^2 \, d(\frac{v-1}{2}) \\
&= -\frac{1}{2} \|W_{j_1}\|^2 + \frac{1}{2} \|W_{j_1}\|^2 = 0.
\end{aligned}$$

ii. Suppose $j_1 < j_2$. Then

$$\begin{aligned}
\langle W_{m_1}, W_{m_2} \rangle &= \int_{\mathbb{R}} W_{2j_1+q_1}(t) W_{2j_2+q_2}(t) \, dt \\
&= \int_0^{\frac{1}{2}} (-1)^{\lfloor \frac{j_1}{2} \rfloor + q_1} W_{j_1}(2t) \cdot (-1)^{\lfloor \frac{j}{2} \rfloor + q_2} W_{j_2}(2t) \, dt \\
&+ \int_{\frac{1}{2}}^1 (-1)^{j_1 + \lfloor \frac{j_1}{2}} W_{j_1}(2t-1) \cdot (-1)^{j_2 + \lfloor \frac{j_2}{2} \rfloor} W_{j_2}(2t-1) \, dt \\
&= (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor + q_1 + q_2} \cdot \frac{1}{2} \int_0^1 W_{j_1}(u) W_{j_2}(u) \, du \\
&+ (-1)^{j_1 + j_2 + \lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} \cdot \frac{1}{2} \int_0^1 W_{j_1}(v) W_{j_2}(v) \, dv \\
&= \left[(-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor + q_1 + q_2} + (-1)^{j_1 + j_2 + \lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} \right] \langle W_{j_1}, W_{j_2} \rangle = 0
\end{aligned}$$

by the induction hypothesis.

Recall that $P(m)$ states that

$$\{W_0, \ldots, W_m\} \text{ is orthogonal in } (L^2(\mathbb{R}), \langle \cdot, \cdot \rangle).$$

Hence even if we have proven $P(m)$ to be true for any $m \in \mathbb{N} \cup \{0\}$,

$$\mathcal{W} \text{ is orthogonal in } (L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$$

has not been directly proven. The subtle difference is easier to observe if we consider the statements

$$\tilde{P}(m) : \{0, \ldots, m\} \text{ is finite}$$

and

$$\mathbb{N} \cup \{0\} \text{ is finite},$$

for which $\tilde{P}(m)$ being true for any $m \in \mathbb{N} \cup \{0\}$ does not imply the truthfulness of the second statement. However, since the orthogonality of $\mathcal{W}$ depends on the orthogonality of pairs of its elements, and each pair of its elements is contained in some $\{W_0 \ldots, W_m\}$, the induction result suffices.

11. (a) $U = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$.

$$\begin{aligned}
\hat{f} &= UfU \\
&= \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}\begin{pmatrix} 3 & 2 & 4 & 4 \\ 4 & -3 & 4 & 0 \\ -2 & -1 & -2 & 3 \\ 4 & 1 & 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix} 9 & -1 & 10 & 5 \\ 5 & 3+4j & 6 & 1-2j \\ -7 & 3 & -6 & 9 \\ 5 & 3-4j & 6 & 1+2j \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix} 23 & -1+6j & 15 & -1-6j \\ 15+2j & 5-2j & 7-2j & -7+2j \\ -1 & -1+6j & -25 & -1-6j \\ 15-2j & -7-2j & 7+2j & 5+2j \end{pmatrix}.
\end{aligned}$$

(b) The submatrix of $\hat{f}$ formed by the four highest frequencies closest to 0 is $\hat{f}' = \frac{1}{16}\begin{pmatrix} 23 & -1+6j & 0 & -1-6j \\ 15+2j & 5-2j & 0 & -7+2j \\ 0 & 0 & 0 & 0 \\ 15-2j & -7-2j & 0 & 5+2j \end{pmatrix}$, whose reconstructed image is

$$\begin{aligned}
(4\bar{U})\hat{f}'(4\bar{U}) &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}\begin{pmatrix} 23 & -1+6j & 0 & -1-6j \\ 15+2j & 5-2j & 0 & -7+2j \\ 0 & 0 & 0 & 0 \\ 15-2j & -7-2j & 0 & 5+2j \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \\
&= \begin{pmatrix} 47 & 49 & 59 & 57 \\ 17 & -17 & 21 & 55 \\ -5 & -27 & -9 & 13 \\ 25 & 39 & 29 & 15 \end{pmatrix}.
\end{aligned}$$

12. (a) i. Refer to DFT of convolution of **Further properties of DFT** in Section 2.3.

ii.

$$
\begin{aligned}
iDFT(MN\hat{f}\odot\hat{g})(k,l) &= MN\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\hat{f}(m,n)\hat{g}(m,n)e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})}\\
&= \frac{1}{MN}\sum_{m,k',k''=0}^{M-1}\sum_{n,l',l''=0}^{N-1} f(k',l')g(k'',l'')e^{2\pi j(\frac{m(k-k'-k'')}{M}+\frac{n(l-l'-l'')}{N})}\\
&= \sum_{k',k''=0}^{M-1}\sum_{l',l''=0}^{N-1} f(k',l')g(k'',l'')\mathbf{1}_{M\mathbb{Z}}(k-k'-k'')\mathbf{1}_{N\mathbb{Z}}(l-l'-l'')\\
&= \sum_{k',k''=0}^{M-1}\sum_{l',l''=0}^{N-1} f(k',l')g(k'',l'')[\delta(k-k'-k'')+\delta(k-k'-k''+M)]\\
&\quad [\delta(l-l'-l'')+\delta(l-l'-l''+N)]\\
&= \sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1} f(k',l')g(k-k',l-l') = f*g(k,l).
\end{aligned}
$$

(b) i.

$$
\widehat{f\odot g}(m,n) = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f(k,l)g(k,l)e^{-2\pi j(\frac{mk}{M}+\frac{nl}{N})},
$$

whereas

$$
\begin{aligned}
\hat{f}*\hat{g}(m,n) &= \sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1}\hat{f}(m',n')\hat{g}(m-m',g-g')\\
&= \frac{1}{M^2N^2}\sum_{m',k,k'=0}^{M-1}\sum_{n',l,l'=0}^{N-1} f(k,l)e^{-2\pi j(\frac{m'k}{M}+\frac{n'l}{N})}g(k',l')e^{-2\pi j(\frac{(m-m')k'}{M}+\frac{(n-n')l'}{N})}\\
&= \frac{1}{M^2N^2}\sum_{m',k,k'=0}^{M-1}\sum_{n'l,l'=0}^{N-1} f(k,l)g(k',l')e^{-2\pi j(\frac{mk'+m'(k-k')}{M}+\frac{nl'+n'(l-l')}{N})}\\
&= \frac{1}{MN}\sum_{k,k'=0}^{M-1}\sum_{l,l'=0}^{N-1} f(k,l)g(k',l')e^{-2\pi j(\frac{mk'}{M}+\frac{nl'}{N})}\delta(k-k')\delta(l-l')\\
&= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f(k,l)g(k,l)e^{-2\pi(\frac{mk}{M}+\frac{nl}{N})} = \widehat{f\odot g}(m,n).
\end{aligned}
$$

ii.

$$
\begin{aligned}
iDFT(\hat{f} * \hat{g})(k,l) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \hat{f} * \hat{g}(m,n) e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})} \\
&= \sum_{m,m'=0}^{M-1}\sum_{n,n'=0}^{N-1} \hat{f}(m',n')\hat{g}(m-m',n-n') e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})} \\
&= \frac{1}{M^2N^2}\sum_{m,m',k',k''=0}^{M-1}\sum_{n,n',l',l''=0}^{N-1} f(k',l')g(k'',l'') \\
&\quad e^{2\pi j(\frac{m(k-k'')+m'(k''-k')}{M}+\frac{n(l-l'')+n'(l''-l')}{N})} \\
&= \sum_{k',k''=0}^{M-1}\sum_{l',l''=0}^{N-1} f(k',l')g(k'',l'') \\
&\quad \mathbf{1}_{M\mathbb{Z}}(k-k'')\mathbf{1}_{M\mathbb{Z}}(k'-k'')\mathbf{1}_{N\mathbb{Z}}(l-l'')\mathbf{1}_{N\mathbb{Z}}(l'-l'') \\
&= f(k,l)g(k,l).
\end{aligned}
$$

(c) i.

$$
\begin{aligned}
\hat{\tilde{f}}(m,n) &= \frac{1}{N^2}\sum_{k,l=0}^{N-1} \tilde{f}(k,l) e^{-2\pi j\frac{mk+nl}{N}} \\
&= \frac{1}{N^2}\sum_{k,l=0}^{N-1} f(l,-k) e^{-2\pi j\frac{mk+nl}{N}},
\end{aligned}
$$

whereas

$$
\begin{aligned}
\tilde{\hat{f}}(m,n) &= \hat{f}(n,-m) \\
&= \frac{1}{N^2}\sum_{k,l=0}^{N-1} f(k,l) e^{-2\pi j\frac{nk-ml}{N}} \\
&= \frac{1}{N^2}\sum_{l'=0}^{N-1}\sum_{k'=1-N}^{0} f(l',-k') e^{-2\pi j\frac{mk'+nl'}{N}} \\
&= \frac{1}{N^2}\sum_{l'=0}^{N-1}\left(f(l',0)e^{-2\pi j\frac{nl'}{N}} + \sum_{k'=1-N}^{0} f(l',-k') e^{-2\pi j\frac{mk'+nl'}{N}}\right) \\
&= \frac{1}{N^2}\sum_{k',l'=0}^{N-1} f(l',-k') e^{-2\pi j\frac{mk'+nl'}{N}} = \hat{\tilde{f}}(m,n).
\end{aligned}
$$

ii.

$$\begin{aligned}
iDFT(\tilde{\hat{f}})(k,l) &= \sum_{m,n=0}^{N-1} \tilde{\hat{f}}(m,n)e^{2\pi j\frac{mk+nl}{N}} \\
&= \sum_{m,n=0}^{N-1} \hat{f}(n,-m)e^{2\pi j\frac{mk+nl}{N}} \\
&= \sum_{m'=0}^{N-1}\sum_{n'=1-N}^{0} \hat{f}(m',n')e^{2\pi j\frac{-n'k+m'l}{N}} \\
&= \sum_{m'=0}^{N-1}\left(\hat{f}(m',0)e^{2\pi j\frac{m'l}{N}} + \sum_{n'=1-N}^{-1}\hat{f}(m',n'+N)e^{2\pi j\frac{-n'k+m'l}{N}}\right) \\
&= \sum_{m',n'=0}^{N-1} \hat{f}(m',n')e^{2\pi j\frac{m'l-n'k}{N}} = f(l,-k).
\end{aligned}$$

(d) WLOG assume $k_0 \in \mathbb{Z}\cap[0,M-1]$ and $l_0 \in \mathbb{Z}\cap[0,N-1]$.

i. Refer to DFT of a shifted image of **Further properties of DFT** in Section 2.3.

ii.

$$\begin{aligned}
iDFT(e^{-2\pi j(\frac{k_0 m}{M}+\frac{l_0 n}{N})}\hat{f})(k,l) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\hat{f}(m,n)e^{2\pi j(\frac{m(k-k_0)}{M}+\frac{n(l-l_0)}{N})} \\
&= f(k-k_0,l-l_0).
\end{aligned}$$

(e) i.

$$\begin{aligned}
\tilde{\hat{f}}(m,n) &= \hat{f}(m-m_0,n-n_0) \\
&= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f(k,l)e^{-2\pi j(\frac{k(m-m_0)}{M}+\frac{l(n-n_0)}{N})} \\
&= DFT(e^{2\pi j(\frac{m_0 k}{M}+\frac{n_0 l}{N})}f)(m,n).
\end{aligned}$$

ii.

$$\begin{aligned}
iDFT(\tilde{\hat{f}})(k,l) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\tilde{\hat{f}}(m,n)e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\hat{f}(m-m_0,n-n_0)e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})} \\
&= \sum_{m'=-m_0}^{M-1-m_0}\sum_{n'=-n_0}^{N-1-n_0}\hat{f}(m',n')e^{2\pi j(\frac{(m'+m_0)k}{M}+\frac{(n'+n_0)l}{N})} \\
&= e^{2\pi j(\frac{m_0 k}{M}+\frac{n_0 l}{N})}f(k,l).
\end{aligned}$$

13. Recall that for any $f \in M_{M\times N}(\mathbb{R})$, $DFT(h*f)(u,v) = MN\,DFT(h)(u,v)DFT(f)(u,v)$; hence $H(u,v) = MN\,DFT(h)(u,v)$.

(a)

$$\begin{aligned}
H_1(u,v) &= \sum_{x=-k}^{k}\sum_{y=-k}^{k}\frac{1}{(2k+1)^2}e^{-2\pi j(\frac{ux}{M}+\frac{vy}{N})} = \frac{1}{(2k+1)^2}\sum_{x=-k}^{k}e^{-2\pi j\frac{ux}{M}}\sum_{y=-k}^{k}e^{-2\pi j\frac{vy}{N}} \\
&= \frac{1}{(2k+1)^2}[1+2\sum_{x=1}^{k}\cos\frac{2\pi ux}{M}][1+2\sum_{y=1}^{k}\cos\frac{2\pi vy}{N}].
\end{aligned}$$

(b) $H_2(u,v)=\frac{r}{r+4}+\frac{1}{r+4}(e^{-2\pi j\frac{u}{M}}+e^{2\pi j\frac{u}{M}}+e^{-2\pi j\frac{v}{N}}+e^{2\pi j\frac{v}{N}})=\frac{r+2(\cos\frac{2\pi u}{M}+\cos\frac{2\pi v}{N})}{r+4}$.

(c)

$$\begin{aligned}H_3(u,v)&=\frac{1}{4}+\frac{1}{8}(e^{-2\pi j\frac{u}{M}}+e^{2\pi j\frac{u}{M}}+e^{-2\pi j\frac{v}{N}}+e^{2\pi j\frac{v}{N}})\\&\quad+\frac{1}{16}(e^{-2\pi j(\frac{u}{M}+\frac{v}{N})}+e^{-2\pi j(\frac{u}{M}-\frac{v}{N})}+e^{-2\pi j(-\frac{u}{M}+\frac{v}{N})}+e^{-2\pi j(-\frac{u}{M}-\frac{v}{N})})\\&=\frac{1}{4}+\frac{1}{4}(\cos\frac{2\pi u}{M}+\cos\frac{2\pi v}{N})+\frac{1}{4}\cos\frac{2\pi u}{M}\cos\frac{2\pi v}{N}\\&=\frac{1}{4}(\cos\frac{2\pi u}{M}+1)(\cos\frac{2\pi v}{N}+1)\\&=\cos^2\frac{\pi u}{M}\cos^2\frac{\pi v}{N}.\end{aligned}$$

(d)

$$\begin{aligned}H_4(u,v)&=-4+e^{-2\pi j\frac{u}{M}}+e^{2\pi j\frac{u}{M}}+e^{-2\pi j\frac{v}{N}}+e^{2\pi j\frac{v}{N}}\\&=-4+2\cos\frac{2\pi u}{M}+2\cos\frac{2\pi v}{N}\\&=-4(\sin^2\frac{\pi u}{M}+\sin^2\frac{\pi v}{N}).\end{aligned}$$

(e)

$$\begin{aligned}H_5(u,v)&=\frac{1}{T}\sum_{t=0}^{T-1}e^{-2\pi j(\frac{atu}{M}+\frac{btv}{N})}\\&=\begin{cases}\frac{1}{T}\cdot\frac{1-e^{-2\pi jT(\frac{au}{M}+\frac{bv}{N})}}{1-e^{-2\pi j(\frac{au}{M}+\frac{bv}{N})}} & \text{if } \frac{au}{M}+\frac{bv}{N}\notin\mathbb{Z},\\ 1 & \text{otherwise,}\end{cases}\\&=\begin{cases}\frac{1}{T}e^{-\pi j(T-1)(\frac{au}{M}+\frac{bv}{N})}\frac{e^{\pi jT(\frac{au}{M}+\frac{bv}{N})}-e^{-\pi jT(\frac{au}{M}+\frac{bv}{N})}}{e^{\pi j(\frac{au}{M}+\frac{bv}{N})}-e^{-\pi j(\frac{au}{M}+\frac{bv}{N})}} & \text{if } \frac{au}{M}+\frac{bv}{N}\notin\mathbb{Z},\\ 1 & \text{otherwise,}\end{cases}\\&=\begin{cases}\frac{1}{T}e^{-\pi j(T-1)(\frac{au}{M}+\frac{bv}{N})}\frac{\sin(\pi T(\frac{au}{M}+\frac{bv}{N}))}{\sin(\pi(\frac{au}{M}+\frac{bv}{N}))} & \text{if } \frac{au}{M}+\frac{bv}{N}\notin\mathbb{Z},\\ 1 & \text{otherwise.}\end{cases}\end{aligned}$$

14.

$$\begin{aligned}&\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}F(m,n)\overline{F(m,n)}\\&=\frac{1}{M^2N^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}f(k,l)e^{-2\pi j(\frac{mk}{M}+\frac{nl}{N})}\sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1}\overline{f(k',l')}e^{2\pi j(\frac{mk'}{M}+\frac{nl'}{N})}\\&=\frac{1}{M^2N^2}\sum_{m,k,k'=0}^{M-1}\sum_{n,l,l'=0}^{N-1}f(k,l)\overline{f(k',l')}e^{2\pi j(\frac{m(k'-k)}{M}+\frac{n(l'-l)}{N})}\\&=\frac{1}{M^2N^2}\sum_{k,k'=0}^{M-1}\sum_{l,l'=0}^{N-1}f(k,l)\overline{f(k',l')}\cdot M\mathbf{1}_{M\mathbb{Z}}(k'-k)\cdot N\mathbf{1}_{N\mathbb{Z}}(l'-l)\\&=\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}|f(k,l)|^2.\end{aligned}$$

15. Let $0\le m,n\le 2^N-1$, and let $h\in\mathcal{S}(\tilde{g})$ such that $h(m,n)=\tilde{g}(m,n)$. Let $h'\in\mathcal{S}(\tilde{g})$ such that $h'=h$ except at (m,n), where $h'(m,n)=0$. Then $\|h'\|_0=\|h\|_0-1$, and

noting the Haar transform matrix H for $2^N \times 2^N$ images is unitary, we have

$$\begin{aligned}
\|iHT(h') - g\|_F^2 &= \|H^T(h' - \tilde{g})H\|_F^2 \\
&= \|h' - \tilde{g}\|_F^2 \\
&= \|h - \tilde{g}\|_F^2 + [\tilde{g}(m,n)]^2 \\
&= \|iHT(h) - g\|_F^2 + e^{2(K - m^2 - n^2)}.
\end{aligned}$$

Hence

$$\begin{aligned}
E(h') &= \|h'\|_0 + \|iHT(h') - g\|_F^2 \\
&= (\|h\|_0 - 1) + (\|iHT(h) - g\|_F^2 + e^{2(K - m^2 - n^2)}) \\
&= E(h) + (e^{2(K - m^2 - n^2)} - 1),
\end{aligned}$$

which is less than $E(h)$ if and only if $m^2 + n^2 > M$.
Vice versa; $E(h) < E(h')$ if and only if $m^2 + n^2 < M$.

Hence h is a minimizer of $E(\cdot)$ over $\mathcal{S}(\tilde{g})$ if and only if $h(m,n) = \tilde{g}(m,n)$ whenever $m^2 + n^2 < M$, and $h(m,n) = 0$ whenever $m^2 + n^2 > M$. Hence

$$h^* = (h^*(m,n)) = \begin{cases} \tilde{g}(m,n) & \text{if } m^2 + n^2 < K \\ 0 & \text{if } m^2 + n^2 \geq K \end{cases}$$

is a minimizer of $E(\cdot)$ over $\mathcal{S}(\tilde{g})$.