# MMAT5390: Mathematical Image Processing

## Assignment 3

Due: April 12 before 1159PM

Please give detailed steps and reasons in your solutions.

1. For this problem, please disregard the definition of the DFT in the lecture notes.

   Consider this alternative definition for the DFT on $N \times N$ images:

   $$\hat{f}(m,n) = DFT(f)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)e^{2\pi j \frac{mk+nl}{N}}.$$

   (a) Show that the inverse DFT (iDFT) is defined by

   $$f(p,q) = iDFT(\hat{f})(p,q) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} \hat{f}(m,n)e^{-2\pi j \frac{pm+qn}{N}}.$$

   (b) Determine the matrix $U$ used to calculate the DFT of an $N \times N$ image, i.e. $\hat{f} = UfU$.

   (c) Show that $U$ is unitary.

2. Let $\hat{f}$ be the discrete Fourier transform of $M \times N$ image $f$.

   Prove that $\hat{f} * \hat{g} = \widehat{f \odot g}$, where $f \odot g(k,l) = f(k,l)g(k,l)$.

3. The even discrete cosine transform (EDCT) on $N \times N$ images is defined by

   $$\hat{f}_{ec}(m,n) = EDCT(f)(m,n) = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)\cos\frac{\pi m(k+\frac{1}{2})}{N}\cos\frac{\pi n(l+\frac{1}{2})}{N}.$$

   (a) Write down the matrix $T_{ec}$ used to calculate the EDCT of an $N \times N$ image, i.e. $\hat{f}_{ec} = T_{ec} f T_{ec}^T$.

   (b) Suppose $N > 1$. Prove that for any $c \in \mathbb{R}$, $cT_{ec}$ is not unitary.

4. Consider a $2N \times 2N$ image $I = (I(m,n))_{-N \leq m,n \leq N-1}$.

   The Butterworth low-pass filter $H$ of squared radius $D_0^2$ and order $n$ is applied on $DFT(I) = (\hat{I}(u,v))_{-N \leq u,v \leq N-1}$ to give $G(u,v)$.

   Suppose $\hat{I}(0,-1) \neq 0$ and $\hat{I}(-2,1) \neq 0$, and

   $$G(0,-1) = \frac{25}{26}\hat{I}(0,-1) \text{ and } G(-2,1) = \frac{1}{2}\hat{I}(-2,1).$$

   Find $D_0^2$ and $n$.

5. Consider a Gaussian high-pass filter

   $$H(u,v) = 1 - exp\left(-\frac{u^2+v^2}{2\sigma^2}\right).$$

   Suppose $H(1,4) = \frac{4}{5}H(-3,5)$. Find $\sigma^2$.

6. The discrete Laplace operator $\Delta$ on a periodically extended $N \times N$ image ($N \geq 3$) can be written as:

$$\Delta f(x,y) = f(x+1,y) + f(x,y+1) + f(x,y-1) + f(x-1,y) - 4f(x,y).$$

Prove that $DFT(\Delta f)(u,v) = H(u,v)F(u,v)$ for some $H \in M_{N\times N}(\mathbb{C})$, where $F = DFT(f)$. Find $H(u,v)$ as a trigonometric polynomial in $\frac{\pi u}{N}$ and $\frac{\pi v}{N}$, i.e. as a polynomial in $\sin\frac{\pi u}{N}$, $\cos\frac{\pi u}{N}$, $\sin\frac{\pi v}{N}$ and $\cos\frac{\pi v}{N}$.

7. Suppose $g \in M_{N\times N}(\mathbb{R})$ is a blurred image capturing a static scene. Assume that $g$ is given by:

$$g(i,j) = \frac{1}{\lambda}\sum_{k=0}^{\lambda-1} f(i-k,j) \text{ for } 0 \leq i,j \leq N-1,$$

where $\lambda \in \mathbb{N} \cap [1,N]$ and $f$ is the underlying image (periodically extended). Show that $DFT(g)(u,v) = H(u,v)DFT(f)(u,v)$ for all $0 \leq u,v \leq N-1$, where $H(u,v)$ is the degradation function in the frequency domain given by:

$$H(u,v) = \begin{cases} \frac{1}{\lambda}\frac{\sin\frac{\lambda\pi u}{N}}{\sin\frac{\pi u}{N}} e^{-\pi j\frac{(\lambda-1)u}{N}} & \text{if } u \neq 0, \\ 1 & \text{if } u = 0. \end{cases}$$