Lecture 9:

Recall: Math. formulation for image blur

observed ↓ ... original ↙

$$g = \underbrace{\tilde{h} * f}_{\text{Blur}} + \underbrace{n}_{\text{noise}}$$

$$\Big\downarrow \text{DFT}$$

$$G(u,v) = \underbrace{c\,\tilde{H}}_{H}(u,v)\,F(u,v) + N(u,v)$$

$$\Big\downarrow \text{iDFT}$$

$$f$$

Image deblurring in the frequency domain: (Assume H is known)

Method 1: Direct inverse filtering

Let $T(u,v) = \dfrac{1}{H(u,v) + \underbrace{\varepsilon \, \mathrm{sgn}(H(u,v))}_{\text{Avoid singularity}}}$ ($\mathrm{sgn}(z) = 1$ if $\mathrm{Re}(z) \geq 0$ and $\mathrm{sgn}(z) = -1$ otherwise)

Compute $\hat{F}(u,v) = G(u,v)\, T(u,v)$.

Find inverse DFT of $\hat{F}(u,v)$ to get an image $\hat{f}(x,y)$.

Good: Simple

Bad: Boast up noise

$$\hat{F}(u,v) = \underbrace{G(u,v)}_{= H(u,v)F(u,v) + N(u,v)} T(u,v) \approx F(u,v) + \frac{N(u,v)}{H(u,v) + \varepsilon \, \mathrm{sgn}(H(u,v))}$$

Note: $H(u,v)$ is big for (u,v) close to $(0,0)$ (Keep low frequencies)

is small for (u,v) far away from $(0,0)$

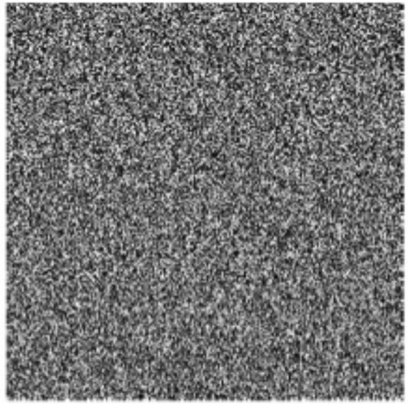
$\therefore \dfrac{N(u,v)}{H(u,v) + \varepsilon \, \mathrm{sgn}(H(u,v))}$ is big for (u,v) far away from $(0,0)$ — Large gain in high frequencies

$\Downarrow$

Boast up noises!!

Original

Blurred image



Direct inverse filtering

Method 2: Modified inverse filtering

Let $B(u,v) = \dfrac{1}{1 + \left(\dfrac{u^2+v^2}{D^2}\right)^n}$ and $T(u,v) = \dfrac{B(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$.

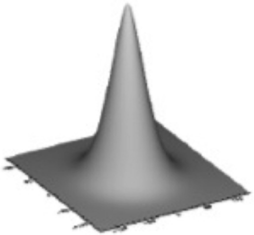
Then define: $\hat{F}(u,v) = T(u,v)\, G(u,v) \approx F(u,v)\, B(u,v) + \dfrac{N(u,v)\, B(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$

$\dfrac{N(u,v)\, B(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))} \approx \dfrac{N(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$ for $(u,v) \approx (0,0)$

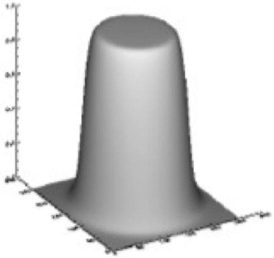
$\dfrac{N(u,v)\, B(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$ is small (as $B(u,v)$ is small) for (u,v) far away from $(0,0)$.

$\dfrac{B(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$ suppresses the high-frequency gain.

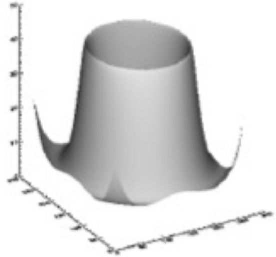
Bad: Has to choose D and n carefully.



$H(u, v)$



$B(u, v)$: D $= 90, n = 8$



Inverse B/H

Original Image $G(u, v)$

Blurred using D $= 90, n = 8$

Restored with a best D and n.

Method 3: Wiener filter

Let $$T(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + \frac{S_n(u,v)}{S_f(u,v)}}$$ where $S_n(u,v) = |N(u,v)|^2$, $S_f(u,v) = |F(u,v)|^2$

If $S_n(u,v)$ and $S_f(u,v)$ are not known, then we let $K = \frac{S_n(u,v)}{S_f(u,v)}$ to get:

$$T(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + K}$$

Let $\hat{F}(u,v) = T(u,v)\,G(u,v)$. Compute $\hat{f}(x,y) =$ inverse DFT of $\hat{F}(u,v)$.

In fact, the Wiener filter can be described as an inverse filtering as follows:

$$\hat{F}(u,v) = \left[\left(\frac{1}{H(u,v)}\right)\left(\frac{|H(u,v)|^2}{|H(u,v)|^2 + K}\right)\right] G(u,v)$$

Behave like "Modified inverse filtering"

≈ 0 if $H(u,v) \approx 0$ (if (u,v) far away from 0)

≈ 1 if $H(u,v)$ is large (if $(u,v) \approx (0,0)$)

What does Wiener filtering do mathematically?

We'll show: Wiener filter minimizes the mean square error:

$$\varepsilon^2(\underbrace{f}_{\text{original}}, \underbrace{\hat{f}}_{\text{Restored}}) = \iint |f(x,y) - \hat{f}(x,y)|^2 \, dx \, dy$$

(We assume the continuous case to avoid complicated indices)

Let $\underbrace{g}_{\text{Observed}} = \underbrace{h}_{\text{degradation}} * \underbrace{f}_{\text{original}} + \underbrace{n}_{\text{noise}}$

Assume that f and n are spatially uncorrelated:

$$0 = \iint f(x,y)\, n(x+r, y+s) \, dx \, dy \quad \text{for all } r, s.$$

Define: $\hat{f}(x,y) = w(x,y) * g(x,y)$ for some $w(x,y)$

(FT of $\hat{f}$ is like: $W(u,v)\, G(u,v)$)

<u>Goal</u>: Find $W(u,v)$ such that $\varepsilon^2(f, \hat{f})$ is minimized.

Recall: $\hat{f}$ is obtained as follows:

<u>Step 1</u>: Let $\hat{F}(u,v) = \underbrace{W(u,v)}_{\text{Filter}}\, G(u,v)$

<u>Step 2</u>: Compute iFT of $\hat{F}$ to get $\hat{f}$

$\therefore \hat{f} = w * g$ for some w.

(Sketch of proof)

We need to use: Parseval Theorem:

$$\varepsilon^2(f,\hat{f}) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y) - \hat{f}(x,y)|^2 \, dx\, dy = C \iint |F(u,v) - \hat{F}(u,v)|^2 \, du\, dv \quad \text{for some constant } C$$

where $F(u,v) = DFT(f)$, $\hat{F}(u,v) = DFT(\hat{f})$

Then: $\hat{F}(u,v) = W(u,v)\, G(u,v)$ (as $\hat{f}(x,y) = iFT(W(u,v)\, G(u,v))$)

So, $\hat{F}(u,v) = W(u,v)\, G(u,v) = W(u,v)\big(H(u,v)\, F(u,v) + N(u,v)\big)$

In other words, $F - \hat{F} = (1 - WH)F - WN$

and $\varepsilon^2(f,\hat{f}) = C \iint |(1-WH)F - WN|^2 \, du\, dv$

Since f and n are spatially uncorrelated, we can show that:

$$\mathcal{E}^2(f, \hat{f}) = \iint |(1-WH)F|^2 + |WN|^2 \, du\, dv$$

$$\left(\iint (1-WH)F\,\overline{W}\,\overline{N} \, du\, dv = 0 \right)$$

Since we look for $w(x,y)$ (hence $W(u,v)$) such that $\mathcal{E}^2$ is minimized, we can regard $\mathcal{E}^2$ is dependent on W.

To minimize $\mathcal{E}^2(W)$, we consider:

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{E}^2(W + tV) = 0 \text{ for all } V.$$

We get: $\iint -(1-\overline{W}\,\overline{H})H|F|^2 V - (1-WH)\overline{H}|F|^2\overline{V} + \overline{W}|N|^2 V + W|N|^2\overline{V} = 0$ for all V.

Put $V = -(1-WH)\overline{H}|F|^2 + W|N|^2$. Then: we have: $\iint \left| -(1-WH)\overline{H}|F|^2 + W|N|^2 \right|^2 du\, dv = 0.$

$$\therefore \quad -(1-WH)\bar{H}|F|^2 + W|N|^2 = 0$$

$$\Downarrow$$

$$W = \frac{\bar{H}}{|H|^2 + |N|^2/|F|^2}.$$