Lecture 8: Image deblurring

Atmospheric turbulence

Motion Blur

Speeding problem

Image deblurring in the frequency domain:

Mathematical formulation of image blurring

Let g be the observed (blurry) image.

Let f be the original (good) image.

Model g as: $g = H(f) + n$

where H is the degradation function/operator and n is the additive noise.

Assumption on H:

1. H is position invariant:

   Let $g(x,y) = H(f)(x,y)$ and let $\tilde{f}(x,y) := f(x-\alpha, y-\beta)$.

   Then: $H(\tilde{f})(x,y) = g(x-\alpha, y-\beta)$

2. Linear: $H(f_1 + f_2) = H(f_1) + H(f_2)$

   $H(\alpha f) = \alpha H(f)$ where α is a scalar multiplication.

3. Linearity can be extended to integral:

$$H\left(\iint \alpha(u,v) f(x-u, y-v)\, du\, dv\right) = \iint \alpha(u,v) H(f)(x-u, y-v)\, du\, dv$$

With the above assumption, consider an impluse signal:

$$\delta(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0) \\ 0 & \text{if } (x,y) \neq (0,0) \end{cases}$$

Then: $f(x,y) = f * \delta(x,y) = \sum_{\alpha=-M/2}^{M/2-1} \sum_{\beta=-N/2}^{N/2-1} f(\alpha,\beta)\,\delta(x-\alpha, y-\beta)$

$$\therefore \quad g(x,y) = H(f)(x,y)$$

$$= \sum_{\alpha=-M/2}^{M/2-1} \sum_{\beta=-N/2}^{N/2-1} f(\alpha,\beta)\, H(\delta)\,(x-\alpha, y-\beta) \quad \text{(by linearity and position-invariant)}$$

$$= \sum_{\alpha=-M/2}^{M/2-1} \sum_{\beta=-N/2}^{N/2-1} f(\alpha,\beta)\, h(x-\alpha, y-\beta) \quad \text{where} \quad h(x,y) = H(\delta)(x,y)$$

$$= f * h\,(x,y)$$

$\therefore$ With the above assumption,

Degradation / Blur = Convolution

Remark:

1. h is called the point spread function

2. $\therefore g(x,y) = h * f(x,y) + n(x,y)$

In the frequency domain,

$$G(u,v) = c\,H(u,v)\,F(u,v) + N(u,v)$$

constant

$\therefore$ Deblurring can be done by:

Compute: $F(u,v) \approx \dfrac{G(u,v)}{cH(u,v)}$ — from observed image — from known degradation

Obtain: $f(x,y) = DFT^{-1}(F(u,v))$

(Does NOT work very well due to noise!)

Examples of degradation function $H(u,v)$

1. Atmospheric turbulence blur:

$$H(u,v) = e^{-k(u^2+v^2)^{5/6}}$$

where k = degree of turbulence

$k = 0.0025$ (severe)

$k = 0.001$ (mild)

$k = 0.00025$ (low turbulence)

2. Out of focus blur:

In the frequency domain, define $H(u,v)$ as the DFT of

$$h(x,y) = \begin{cases} 1 & \text{if } x^2+y^2 \le D_0^2 \\ 0 & \text{otherwise} \end{cases}$$

In some situations, a simple model:

$$H(u,v) = \begin{cases} 1 & \text{if } u^2+v^2 \le D_0^2 \\ 0 & \text{otherwine} \end{cases}$$

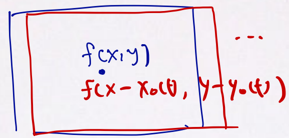
(Usually not accurate)

3. Uniform Linear Motion Blur

Assume $f(x,y)$ undergoes planar motion during acquisition.
(original)

Let $(x_0(t), y_0(t))$ be the motion components in the x- and y-directions
time

Let T be the total exposure time.



The observed image is given by:

$$g(x,y) = \int_0^T f(x - x_0(t), y - y_0(t))\, dt$$

Now, let $G(u,v) = DFT(g)(u,v)$, then:

$$G(u,v) = \frac{1}{N^2} \sum_x \sum_y g(x,y)\, e^{-j\frac{2\pi}{N}(ux+vy)}$$

$$= \frac{1}{N^2} \sum_x \sum_y \int_0^T f(x - x_0(t), y - y_0(t))\, dt\; e^{-j\frac{2\pi}{N}(ux+vy)}$$

$$= \int_0^T \left(\sum_x \sum_y f(x - x_0(t), y - y_0(t))\, e^{-j\frac{2\pi}{N}(ux+vy)} \right) dt$$

Recall that $DFT(f(x-x_0, y-y_0)) = F(u,v)\, e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))}$.

We have: $$G(u,v) = \int_0^T \left[F(u,v)\, e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))} \right] dt$$

$$= F(u,v) \int_0^T e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))}\, dt$$

$$= F(u,v)\, H(u,v)$$

$\therefore$ Degradation function in the frequency domain is given by:

$$H(u,v) = \int_0^T e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))}\, dt$$

(Speeding problem !!)

Example: Suppose the camera is moving left horizontally with a constant speed c.

That is, the image at time t is given by:

$$I^*(x,y) = I(x, y-ct)$$

Then: the degradation function is given by:

$$H(u,v) = \int_0^T e^{-j\frac{2\pi}{N}(v(ct))}\,dt$$

<u>Remark:</u> Once the degradation function is known, the original image can be restored by: $\text{IDFT}\left(\frac{G(u,v)}{H(u,v)}\right)$ (given that there's no noise)

What if there is noise??

Image deblurring in the frequency domain: (Assume H is known)

Method 1: Direct inverse filtering

Let $T(u,v) = \dfrac{1}{cH(u,v) + \underbrace{\varepsilon\, \mathrm{sgn}(H(u,v))}_{\text{Avoid singularity}}}$ ($\mathrm{sgn}(z) = 1$ if $\mathrm{Re}(z) \geq 0$ and $\mathrm{sgn}(z) = -1$ otherwise)

Compute $\hat{F}(u,v) = G(u,v)\, T(u,v)$.

Find inverse DFT of $\hat{F}(u,v)$ to get an image $\hat{f}(x,y)$.

Good: Simple

Bad: Boast up noise

$$\hat{F}(u,v) = \underbrace{G(u,v)}_{\substack{\shortparallel \\ cH(u,v)F(u,v) + N(u,v)}}\, T(u,v) \approx F(u,v) + \frac{N(u,v)}{cH(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$$

Note: $H(u,v)$ is big for (u,v) close to $(0,0)$ (Keep low frequencies)

is small for (u,v) far away from $(0,0)$

$\therefore \dfrac{N(u,v)}{cH(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$ is big for (u,v) far away from $(0,0)$

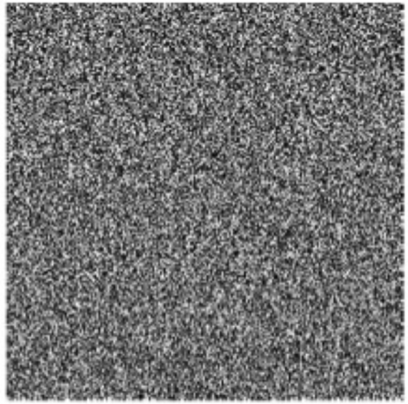
Large gain in high frequencies

$\Downarrow$

Boast up noises!!

Original

Blurred image



Direct inverse filtering

Method 2: Modified inverse filtering

Let $B(u,v) = \dfrac{1}{1+\left(\dfrac{u^2+v^2}{D^2}\right)^n}$ and $T(u,v) = \dfrac{B(u,v)}{cH(u,v)+\varepsilon\,\mathrm{sgn}(H(u,v))}$.

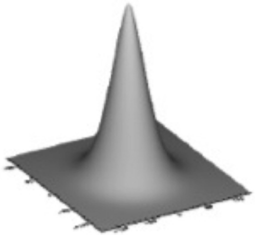
Then define: $\hat{F}(u,v) = T(u,v)\,G(u,v) \approx F(u,v)\,B(u,v) + \dfrac{N(u,v)\,B(u,v)}{cH(u,v)+\varepsilon\,\mathrm{sgn}(H(u,v))}$

$\dfrac{N(u,v)\,B(u,v)}{H(u,v)+\varepsilon\,\mathrm{sgn}(H(u,v))} \approx \dfrac{N(u,v)}{cH(u,v)+\varepsilon\,\mathrm{sgn}(H(u,v))}$ for $(u,v) \approx (0,0)$

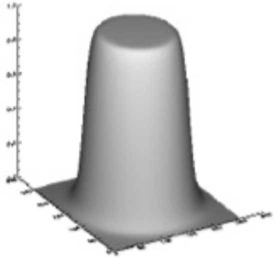
$\dfrac{N(u,v)\,B(u,v)}{cH(u,v)+\varepsilon\,\mathrm{sgn}(H(u,v))}$ is small (as $B(u,v)$ is small) for (u,v) far away from $(0,0)$.

$\dfrac{B(u,v)}{cH(u,v)+\varepsilon\,\mathrm{sgn}(H(u,v))}$ suppresses the high-frequency gain.

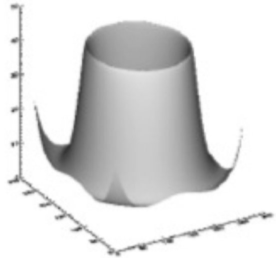
Bad: Has to choose D and n carefully.



$H(u,v)$



$B(u,v)$: $\mathsf{D} = 90, n = 8$



Inverse B/H

Original Image $G(u,v)$

Blurred using $\mathsf{D} = 90, n = 8$

Restored with a best D and n.

Method 3: Wiener filter

Let $T(u,v) = \dfrac{\overline{H(u,v)}}{|H(u,v)|^2 + \frac{S_n(u,v)}{S_f(u,v)}}$ where $S_n(u,v) = |N(u,v)|^2$

$S_f(u,v) = |F(u,v)|^2$

If $S_n(u,v)$ and $S_f(u,v)$ are not known, then we let $K = \dfrac{S_n(u,v)}{S_f(u,v)}$ to get:

$$T(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + K}$$

Let $\hat{F}(u,v) = T(u,v)\,G(u,v)$. Compute $\hat{f}(x,y) =$ inverse DFT of $\hat{F}(u,v)$.

In fact, the Wiener filter can be described as an inverse filtering as follows:

$$\hat{F}(u,v) = \left(\frac{1}{cH(u,v)}\right)\left(\frac{|cH(u,v)|^2}{|cH(u,v)|^2 + K}\right) G(u,v)$$

Behave like "Modified inverse filtering"

≈ 0 if $H(u,v) \approx 0$ (if (u,v) far away from 0)

≈ 1 if $H(u,v)$ is large (if $(u,v) \approx (0,0)$)

What does Wiener filtering do mathematically?

We'll show: Wiener filter minimizes the mean square error:

$$\mathcal{E}^2(\underset{\text{original}}{f}, \underset{\text{Restored}}{\hat{f}}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y) - \hat{f}(x,y)|^2 \, dx \, dy$$

(We assume the continuous case to avoid complicated indices)