Lecture 9: More about Image deblurring

Method 4: Constrained least square filtering

Disadvantages of Wiener's filter:

① $|N(u,v)|^2$ and $|F(u,v)|^2$ must be known / guessed

② Constant estimation of ratio is not always suitable

Goal: Consider a least square minimization model.

Let $g = \underset{\substack{\uparrow \\ \text{degradation}}}{h} * f + n \nwarrow \text{noise}$

In matrix form, $\underset{\substack{\| \\ \mathcal{S}(g)}}{\vec{g}} = \underset{\substack{\uparrow \\ \text{transformation matrix of } h*f \\ (\text{or } \vec{f})}}{D} \underset{\substack{\| \\ \mathcal{S}(f)}}{\vec{f}} + \underset{\substack{\| \\ \mathcal{S}(n)}}{\vec{n}}$ $\quad \vec{g}, \vec{f}, \vec{n} \in \mathbb{R}^{N^2}$, $D \in M_{N^2 \times N^2}$

Given $\vec{g}$, we need to find an estimation of $\vec{f}$ such that it minimizes:

$$E(f) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} |\nabla^2 f(x,y)|^2$$

subject to the constraint:

$$\|\vec{g} - D\vec{f}\|^2 = \varepsilon$$

- $\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} |\nabla^2 f(x,y)|^2$ ← Denoise
- $\|\vec{g} - D\vec{f}\|^2 = \varepsilon$ ← Deblur

In the discrete case, we can estimate:

$$\nabla^2 f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)$$

Taylor expansion:

$$\frac{\partial^2 f(x,y)}{\partial x^2} \approx \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2} \xrightarrow{\text{Put } h=1} \nabla^2 f(x,y) = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)(x,y)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) \approx \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2}$$

More generally, $\nabla^2 f = p * f$ ← discrete convolution

where

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ & 1 & -4 & 1 & \\ \vdots & & 1 & & \vdots \\ 0 & & \cdots & & 0 \end{pmatrix}$$

Remark: $\|\underset{\overset{\shortparallel}{\|\vec{n}\|^2}}{\vec{g} - D\vec{f}}\|^2 = \varepsilon$ means we Allow some fixed level of noise.

Assume $\mathcal{S}(p * f) = L\vec{f}$

transformation matrix representing the convolution with p.

Then: $E(\vec{f}) = (L\vec{f})^T(L\vec{f})$

We will prove:

Theorem: The constrained least square problem has the optimal solution in the spatial domain that satisfies:

$$(D^TD + \gamma L^TL)\vec{f} = D^T\vec{g}$$

for some suitable parameter γ.

In the frequency domain,

$$\hat{F}(u,v) := DFT(f)(u,v) = \frac{1}{N^2}\,\frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma\,|P(u,v)|^2}\,G(u,v)$$

$(H = DFT(h)$; $G(u,v) = DFT(g)$; $P(u,v) = DFT(p)$ where

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}\quad)$$

Remark: Constrained least square filtering:

$$T(u,v) = \frac{1}{N^2} \frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma |P(u,v)|^2}$$

Let $\tilde{F}(u,v) = T(u,v)\, G(u,v)$

Compute Inverse DFT of $\tilde{F}(u,v)$.

Sketch of proof:

Recall: our problem is to minimize:

$$\vec{f}^T L^T L \vec{f} \quad \text{subject to} \quad \underset{(\vec{g} - D\vec{f})^T (g - D\vec{f})}{\underset{\parallel}{\|\vec{g} - D\vec{f}\|^2}} = \varepsilon$$

From calculus, the minimizer must satisfy:

$$\mathcal{D} = \frac{\partial}{\partial \vec{f}} \left(\vec{f}^T L^T L \vec{f} + \lambda (\vec{g} - D\vec{f})^T (\vec{g} - D\vec{f}) \right) = 0 \quad \text{for}$$

where $\vec{f} = (f_1, f_2, \dots, f_i, \dots, f_{N^2})^T$ and λ is the Lagrange's multiplier.

Here, $\frac{\partial K}{\partial \vec{f}} = \left(\frac{\partial K}{\partial f_1}, \frac{\partial K}{\partial f_2}, \dots, \frac{\partial K}{\partial f_{N^2}} \right)^T$

Easy to check:

- $\frac{\partial (\vec{f}^T \vec{a})}{\partial \vec{f}} = \vec{a}$
- $\frac{\partial (\vec{b}^T \vec{f})}{\partial \vec{f}} = \vec{b}$
- $\frac{\partial (\vec{f}^T A \vec{f})}{\partial \vec{f}} = (A + A^T) \vec{f}$

$$\vec{f}^{T}\vec{a} = (f_1, f_2, \dots, f_n)\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 f_1 + a_2 f_2 + \dots + a_n f_n$$

$$\frac{\partial \vec{f}^{T}\vec{a}}{\partial f_j} = a_j$$

$$\therefore \frac{\partial \vec{f}^{T}\vec{a}}{\partial \vec{f}} \overset{\text{def}}{=} \left(\frac{\partial \vec{f}^{T}\vec{a}}{\partial f_1}, \dots, \frac{\partial \vec{f}^{T}\vec{a}}{\partial f_n} \right) = (a_1, a_2, \dots, a_n)$$

etc. . ..

$$\therefore\ D = 0 \Rightarrow (2L^TL)\vec{f} + \lambda(-D^T\vec{g} - D^T\vec{g} + 2D^TD\vec{f}) = 0$$

$$\Rightarrow (D^TD + \gamma L^TL)\vec{f} = D^T\vec{g}$$ where $\gamma = \frac{1}{\lambda}$ and λ is the Lagrange's multiplier.

Parameter γ can be determined by direct substitution into the equation:

$$(\vec{g} - D\vec{f})^T(\vec{g} - D\vec{f}) = \varepsilon.$$

Now, we'll consider the frequency domain.

Note that D and L are transformation matrix of convolution.

∴ D and L are block-circulant.

Some facts about circulant matrix:

Recall: A matrix is block-circulant if

$$H = \begin{pmatrix} H_0 & H_{M-1} & H_{M-2} & \cdots & H_1 \\ H_1 & H_0 & H_{M-1} & \cdots & H_2 \\ H_2 & H_1 & H_0 & \cdots & H_3 \\ \vdots & \vdots & \vdots & & \vdots \\ H_{M-1} & H_{M-2} & H_{M-3} & \cdots & H_0 \end{pmatrix}$$

(each H_i is circulant)

A matrix $\mathcal{C}$ is circulant if:

$$\mathcal{C} = \begin{pmatrix} d_0 & d_{M-1} & d_{M-2} & \cdots & d_1 \\ d_1 & d_0 & d_{M-1} & \cdots & d_2 \\ d_2 & d_1 & d_0 & \cdots & d_3 \\ \vdots & \vdots & \vdots & & \vdots \\ d_{M-1} & d_{M-2} & d_{M-3} & \cdots & d_0 \end{pmatrix}$$

Eigenvalues / Eigenvectors of circulant $\mathcal{C}$

Let $\mathcal{C} = \begin{pmatrix} d(0) & d(M-1) & \cdots & d(1) \\ d(1) & d(0) & \cdots & d(2) \\ \vdots & \vdots & \cdots & \vdots \\ d(M-1) & d(M-2) & \cdots & d(0) \end{pmatrix}$ be a circulant matrix. Then the eigenvalues of $\mathcal{C}$ is given by:

$$\underline{\lambda(k) = d(0) + d(1)e^{\frac{2\pi j}{M}(M-1)k} + d(2)e^{\frac{2\pi j}{M}(M-2)k} + \cdots + d(M-1)e^{\frac{2\pi j}{M}k}}$$

(eigenvalue)

where $k = 0, 1, 2, \cdots M-1$.

Its associated eigenvector is given by:

$$\boxed{\vec{w}(k) = \begin{pmatrix} 1 \\ e^{\frac{2\pi j}{M}k} \\ e^{\frac{2\pi j}{M}2k} \\ \vdots \\ e^{\frac{2\pi j}{M}(M-1)k} \end{pmatrix}}$$

(eigenvector)

Diagonalization of block-circulant matrix

D (transformation matrix of $h * f$)

Let H be the block-circulant matrix as defined above. Define a matrix with elements:

$$W_N(k,n) := \frac{1}{\sqrt{N}} exp\left(\frac{2\pi j}{N} kn\right) \qquad 0 \leq n \leq N-1$$

Consider the **Kronecker product** $\otimes$ of W_N with itself:

$$W := W_N \otimes W_N$$

Recall that: Kronecker product is defined as:

The **Kronecker product** of two matrices are given by:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & \cdots & a_{2N}B \\ \vdots & \vdots & & \vdots \\ a_{N1}B & a_{N2}B & \cdots & a_{NN}B \end{pmatrix}$$

$A = (a_{ij})_{0 \leq i,j \leq N-1}$

$B = (b_{ij})_{0 \leq i,j \leq N-1}$

W^{-1} can be easily computed!

Easy to check: $W^{-1} = W_N^{-1} \otimes W_N^{-1}$ where:

$$W_N^{-1}(k,n) := \frac{1}{\sqrt{N}} exp\left(-\frac{2\pi j}{N} kn\right) \qquad 0 \leq n \leq N-1$$

Let

$$\Lambda(k,i) = \begin{cases} N^2 \ddot{H}\left(\text{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where H = DFT of the point spread function h, $\left\lfloor \frac{k}{N} \right\rfloor$ = largest integer smaller than or equal to $\frac{k}{N}$ and $\text{mod}_N(k) = k(\text{mod } N)$ (e.g. $10(\text{mod } 3) = 1$)

Then, we can show that $H = W\Lambda W^{-1}$ and $H^{-1} = W\Lambda^{-1}W^{-1}$.

Also, $H^T = W\Lambda^* W^{-1}$. (Λ^* is the complex conjugate of Λ)

By direct calculation, it is easy to check that $W^{-1}\vec{g} = N\zeta(G)$ where $G = DFT(g)$.

Using the fact that both D and L are block-circulant, we can check that:

$$D = W\Lambda_D W^{-1}, D^T = W\Lambda_D^* W^{-1}, L = W\Lambda_L W^{-1}, L^T = W\Lambda_L^* W^{-1}$$

where W is invertible and Λ_D, Λ_L are diagonal matrices.

Also,

$$\Lambda_D(k,i) = \begin{cases} N^2\ddot{H}\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where $\ddot{H} = DFT(h)$.

and

$$\Lambda_L(k,i) = \begin{cases} N^2 P\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

$P = DFT(p)$;

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}$$

$$\Lambda_D = N^2 \begin{pmatrix} H(0,0) & & & & & & & \\ & H(1,0) & & & & & & \\ & & \ddots & & & & & \\ & & & H(N-1,0) & & & & \\ & & & & H(0,1) & & & \\ & & & & & H(1,1) & & \\ & & & & & & \ddots & \\ & & & & & & & H(N-1,1) & \\ & & & & & & & & \ddots \end{pmatrix}$$

1st col of DFT(h)

2nd col of DFT(h)

Combining these information and substitute into the "governing" equation :

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g} \, ,$$

We get :

$$\cancel{W}(\Lambda_{\underset{\uparrow}{D}}^* \Lambda_{\underset{\uparrow}{D}} + \gamma \Lambda_L^* \Lambda_{\underset{\uparrow}{L}}) W^{-1} \vec{f} = \cancel{W} \Lambda_D^* W^{-1} \vec{g}$$

entries given by DFT(h)

entries given by DFT(p)

Combining these information and substitute into the "governing" equation :

$$(D^T D + \gamma L^T L)\vec{f} = H^T \vec{g} ,$$

We get:
$$\cancel{W}(\Lambda_D^* \Lambda_D + \gamma \Lambda_L^* \Lambda_L) W^{-1}\vec{f} = \cancel{W}\Lambda_D^* W^{-1}\vec{g}$$

We can check that:

①
$$\boxed{\Lambda_D^* \Lambda_D = \begin{pmatrix} N^4|H(0,0)|^2 & & & & \\ & N^4|H(1,0)|^2 & & & \\ & & \ddots & & \\ & & & N^4|H(N-1,0)|^2 & \\ & & & & \ddots \\ & & & & & N^4|H(N-1,N-1)|^2 \end{pmatrix}}$$

$$H = DFT(h)$$

②
$$\boxed{\Lambda_L^* \Lambda_L = \begin{pmatrix} N^4|P(0,0)|^2 & & & & \\ & N^4|P(1,0)|^2 & & & \\ & & \ddots & & \\ & & & N^4|P(N-1,0)|^2 & \\ & & & & \ddots \\ & & & & & N^4|P(N-1,N-1)|^2 \end{pmatrix}}$$

$$P = DFT(p)$$

③ $\boxed{W^{-1}\vec{f} = N\mathcal{S}(F), W^{-1}\vec{g} = N\mathcal{S}(G)}$ where $F = DFT(f), G = DFT(g)$.

Suppose D is the transformation matrix representing the convolution with h.

(In other words, if $g = h * f$, then: $\underset{\mathbb{R}^{N^2}}{\underset{\rotatebox{90}{$\in$}}{\vec{g}}} = \underset{M_{N^2 \times N^2}}{\underset{\rotatebox{90}{$\in$}}{D}} \vec{f} \in \mathbb{R}^{N^2}$)

Let $H = \mathrm{DFT}(h) \in M_{N \times N}$

Diagonalization of D:

$$D = WN^2 \underbrace{\left[\begin{pmatrix} H(0,0) &&&&&&&&&& \\ & H(1,0) &&&&&&&&& \\ && \ddots &&&&&&&& \\ &&& H(N-1,0) &&&&&&& \\ &&&& H(0,1) &&&&&& \\ &&&&& \ddots &&&&& \\ &&&&&& H(N-1,1) &&&& \\ &&&&&&& \ddots &&& \\ &&&&&&&& H(0,N-1) && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& H(N-1,N-1) \end{pmatrix}\right]}_{\text{Stack } H \text{ to form the diagonal matrix.}} W^{-1}$$

Fact 2:

$$W^{-1}\vec{f} = \begin{pmatrix} F(0,0) \\ \vdots \\ F(N-1,0) \\ F(0,1) \\ \vdots \\ F(N-1,1) \\ \vdots \\ F(0,N-1) \\ \vdots \\ F(N-1,N-1) \end{pmatrix} = \mathcal{S}(F) \quad \text{where } F = \mathrm{DFT}(f).$$

Example: Assume that :

$$G = \begin{pmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix} \quad \text{and} \quad W_3^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & exp\left(-\frac{2\pi j}{3}\right) & exp\left(-\frac{2\pi j}{3}2\right) \\ 1 & exp\left(-\frac{2\pi j}{3}2\right) & exp\left(-\frac{2\pi j}{3}\right) \end{pmatrix}$$

Then:

$$W^{-1} = W_3^{-1} \otimes W_3^{-1} = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2}
\end{pmatrix}$$

$$W^{-1}\vec{g} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\ 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\ 1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\ 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} \\ 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 \\ 1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\ 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 \\ 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} \end{pmatrix}\begin{pmatrix} g_{00} \\ g_{10} \\ g_{20} \\ g_{01} \\ g_{11} \\ g_{21} \\ g_{02} \\ g_{12} \\ g_{22} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} g_{00} + g_{10} + g_{20} + g_{01} + g_{11} + g_{21} + g_{02} + g_{12} + g_{22} \\ g_{00} + g_{10}e^{-\frac{2\pi j}{3}} + g_{20}e^{-\frac{2\pi j}{3}2} + g_{01} + g_{11}e^{-\frac{2\pi j}{3}} + g_{21}e^{-\frac{2\pi j}{3}2} + g_{02} + g_{12}e^{-\frac{2\pi j}{3}} + g_{22}e^{-\frac{2\pi j}{3}2} \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}$$

$= 3^2 G(0,0)$

$= 3^2 G(1,0)$

$\vdots$

$G = DFT(g)$

$\therefore \quad W^{-1}\vec{g} = 3\,S(G)$

Combining all these, we get for every (u,v),

$$N^4[|H(u,v)|^2 + \gamma |P(u,v)|^2] N F(u,v) = N^2 \overline{H(u,v)} N G(u,v)$$

$$\Rightarrow \boxed{N^2 \frac{|H(u,v)|^2 + \gamma |P(u,v)|^2}{\overline{H(u,v)}} F(u,v) = G(u,v)}$$

Summary: Constrained least square filtering minimizes:

$$E(\vec{f}) = (L\vec{f})^T (L\vec{f})$$

subject to the constraint that:

$$\| \underbrace{\vec{g} - D\vec{f}}_{\vec{n}} \|^2 = \varepsilon$$

(allow fixed amount of noise)