MMAT5390 Mathematical Image Processing
Midterm Examination

You have to answer all five questions. The total score is **100**.
Please show your steps unless otherwise stated.

1. (**20pts**) Determine if the following PSFs of linear transformations on $N \times N$ square images are separable and/or shift-invariant (with h_s being N-periodic in both arguments) by observing their corresponding transformation matrices. Please justify your answers with detailed explanations.

(a) $$H = \begin{pmatrix} 2 & 3 & 1 & 6 & 9 & 3 & 4 & 6 & 2 \\ 1 & 2 & 3 & 3 & 6 & 9 & 2 & 4 & 6 \\ 3 & 1 & 2 & 9 & 3 & 6 & 6 & 2 & 4 \\ 4 & 6 & 2 & 2 & 3 & 1 & 6 & 9 & 3 \\ 2 & 4 & 6 & 1 & 2 & 3 & 3 & 6 & 9 \\ 6 & 2 & 4 & 3 & 1 & 2 & 9 & 3 & 6 \\ 6 & 9 & 3 & 4 & 6 & 2 & 2 & 3 & 1 \\ 3 & 6 & 9 & 2 & 4 & 6 & 1 & 2 & 3 \\ 9 & 3 & 6 & 6 & 2 & 4 & 3 & 1 & 2 \end{pmatrix};$$

(b) $$H = \begin{pmatrix} e^{-1} & e^{-2} & e^{-3} & 0 & 0 & 0 & e^{-2} & e^{-3} & e^{-4} \\ e^{-4} & e^{-5} & e^{-6} & 0 & 0 & 0 & e^{-5} & e^{-6} & e^{-7} \\ e^{-7} & e^{-8} & e^{-9} & 0 & 0 & 0 & e^{-8} & e^{-9} & e^{-10} \\ e^{-2} & e^{-3} & e^{-4} & e^{-1} & e^{-2} & e^{-3} & 0 & 0 & 0 \\ e^{-5} & e^{-6} & e^{-7} & e^{-4} & e^{-5} & e^{-6} & 0 & 0 & 0 \\ e^{-8} & e^{-9} & e^{-10} & e^{-7} & e^{-8} & e^{-9} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-2} & e^{-3} & e^{-4} & e^{-1} & e^{-2} & e^{-3} \\ 0 & 0 & 0 & e^{-5} & e^{-6} & e^{-7} & e^{-4} & e^{-5} & e^{-6} \\ 0 & 0 & 0 & e^{-8} & e^{-9} & e^{-10} & e^{-7} & e^{-8} & e^{-9} \end{pmatrix};$$

(c) $$H = \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & \sqrt{2} & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 & 0 & \sqrt{6} & 0 & 3 \\ \sqrt{6} & 2 & 0 & 0 & 0 & 0 & 3 & \sqrt{6} & 0 \\ 0 & \sqrt{6} & 2 & 0 & 0 & 0 & 0 & 3 & \sqrt{6} \\ 0 & 0 & 0 & \sqrt{10} & 0 & \sqrt{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{15} & \sqrt{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{15} & \sqrt{10} & 0 & 0 & 0 \end{pmatrix};$$

(d) $$H = \begin{pmatrix} \cos x & \cos 2x & \sin x & \sin 2x \\ \cos 2x & \cos x & \sin 2x & \sin x \\ \sin x & \sin 2x & \cos x & \cos 2x \\ \sin 2x & \sin x & \cos 2x & \cos x \end{pmatrix}, \text{ where } x \in (0, \pi/4);$$

(e) $$H = \begin{pmatrix} a & 0 & d & e \\ b & c & 0 & f \\ d & e & a & 0 \\ 0 & f & b & c \end{pmatrix}, \text{ where } a, b, c, d \in \mathbb{N}\backslash\{0\};$$

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2. (**20pts**) Consider a 4×4 image of the following form:

$$I = \begin{pmatrix} a & a & b & b \\ c & c & d & d \\ e & e & f & f \\ g & g & h & h \end{pmatrix}$$

where a, b, c, d, e, f, g, h are given by your birthday. For example, if your birthday is 13 October, 1990, then $abcdefgh = 19901013$. If your birthday is 2 March, 1991, then $abcdefgh = 19910302$.

(a) When is your birthday? (Hope you don't mind, I may buy you a birthday cake if possible)

(b) Find the Walsh transform matrix W for a 4×4 image, i.e. the matrix such that the Walsh transform of f is WfW^T. Please show all your steps.

(c) Verify that W is orthogonal.

(d) Compute the Walsh transform of I.

3. (**20pts**) This question is about singular value decomposition.

(a) Consider

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

i. Compute $A^T A$. Find the eigenvalues of $A^T A$. Please show all your steps.

ii. Compute the singular value decomposition of A. Please show all your steps.

iii. Write A as a linear combination of eigen-images.

iv. Find an image $\tilde{A}$ with rank 2 such that $||\tilde{A} - A||_F$ is equal to 2 ($||\cdot||_F$ is the Frobenius norm). Please explain your answer with details.

(b) (**A bit challenging**) Consider a $N \times N$ image, where $N > 100000$. Suppose the singular value decomposition of I is given by $I = V \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_N \end{pmatrix} V^T$. It is known that V is given by:

$$V = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{1,1} & \cdots & a_{1,N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{N-3,1} & \cdots & a_{N-3,N-2} \end{pmatrix}$$

Now, suppose the image I is corrupted by noise $n \in M_{N\times N}(\mathbb{R})$ at the upper left corner. In other words, the noisy image is given by $\tilde{I} = I + n$, where n is given by:

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$$n = \begin{pmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & 0 & \cdots & 0 \\ \epsilon_2 & \epsilon_1 & \epsilon_3 & 0 & \cdots & 0 \\ \epsilon_3 & \epsilon_3 & \epsilon_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Provide that $\epsilon_3 = \frac{\epsilon_1+\epsilon_2}{2}$.
Compute the singular value decomposition of $\tilde{I}$ in terms of $V, \epsilon_1, \epsilon_2, \epsilon_3$ and σ_i's. Please explain your answer with details.

4. (**20pts**) Let $g = (g(k,l))_{0\leq k,l\leq N-1}$ be an $N \times N$ image, and denote its reflection about $(\frac{-1}{2}, \frac{-1}{2})$ by $\tilde{g} = (\tilde{g}(k,l))_{-N\leq k\leq -1, -N\leq l\leq -1}$. That is,

$$\tilde{g}(k,l) = g(-1-k, -1-l) \text{ for } -N \leq k \leq -1 \text{ and } -N \leq l \leq -1.$$

Prove that the discrete Fourier transform (DFT) of $\tilde{g}$ is given by:

$$DFT(\tilde{g})(m,n) = e^{2\pi j \frac{m+n}{N}} \hat{g}(-m,-n),$$

where $\hat{g}$ is the DFT of g. Please show all your steps.

5. (**20pts**) Consider a $N \times N$ image $I = (I(k,l))_{0\leq k,l\leq N-1}$. Let $\hat{I} = (\hat{I}(m,n))_{0\leq m,n\leq N-1}$ be the discrete Fourier transform of I. Assume that N is even and $N > 10000$.

   (a) Explain why the middle region of $\hat{I}$ can be regarded as the high-frequency region and the four corners can be regarded as the low frequency region. Please explain in full details to demonstrate your understanding of the concept.

   (b) (**Challenging**) Let A and B be $N \times N$ matrices defined as follows:

$$A = \begin{pmatrix} 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & & & & \\ & & & D & & & \\ & & & & 1 & 1 & 1 \\ & & & & 1 & 1 & 1 \\ & & & & 1 & 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} & & & & 1 & 1 & 1 \\ & & & & 1 & 1 & 1 \\ & & & & 1 & 1 & 1 \\ & & & E & & & \\ 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & & & & \end{pmatrix}$$

   where D is a $(N-6) \times (N-6)$ real matrix and E is a $(N-6) \times (N-6)$ zero matrix. Note that entries of A and B without indicated values are assumed to be zero. Let $H = \frac{1}{3^{N-7}} A^{N-6} + B$. Consider $\tilde{I}(m,n) = H(m,n)\hat{I}(m,n)$ for $0 \leq m,n \leq N-1$. Prove that $\tilde{I}$ represents a low-pass filtering if $D = VJV^{-1}$, where V is an invertible $(N-6) \times (N-6)$ real matrix and J is given by

$$J = \begin{pmatrix} 0 & a_1 & & & \\ & 0 & a_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & a_{N-7} \\ & & & & 0 \end{pmatrix}$$

   Here, $a_1, ..., a_{N-7}$ are non-zero real numbers.

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